A Shot-noise Approach to Decay-Surge Collective Population Models

Thierry E. Huillet

Laboratoire de Physique Théorique et Modélisation, CY Cergy Paris Université, CNRS UMR-8089, 2 avenue Adolphe-Chauvin, 95302 Cergy-Pontoise, France

ABSTRACT
Shot-noise models deal with the cumulative output of a system whose input is subject to a random Poisson succession of equally distributed impulses or shots, each followed by some attenuation dynamics. With population dynamics in mind, we study the cases when the attenuation dynamics are either given by some ad hoc attenuation function or by some nonlinear ordinary dynamical system or by a (sub-)critical branching process. In the three cases, an interesting issue concerning extinction and idle periods is when the overlapping subpopulations can go extinct in finite time.

1. Introduction

Piecewise-deterministic Markovian decay/surge and growth/collapse population processes were recently investigated in (Goncalves et al. 2020 and Goncalves et al. 2022). We herewith study alternative decay/surge continuous-time shot-noise models, now describing the current accumulation of the declining contributions of overlapping subpopulations (each triggered by random initial independent shot conditions) which occurred at the random times of a Poisson process in the past.

Shot-noise or Schottky noise consists of a model of discontinuous noise pertaining to linearly filtered continuous-time physical systems conveyed by pulses. Its discontinuous nature stems from the discreteness of quanta triggering local flows including electrons or photons in engineering systems, packets in communication systems (see e.g. (Snyder 1991), including for historical background, and (Parzen 1962), but also immigrants in population dynamics, etc... It is an ubiquitous process in Nature. When there is an attenuation process following each shot, we shall speak of decay/surge shot-noise overlapping population models.

Decay/surge shot-noise models have applications in wear processes that need

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recurrent adjunction of fresh material for maintenance (reliability for maintenance). Wear being here the damaging, gradual removal or deformation of material at solid surfaces, caused by erosion (mechanical) or corrosion (chemical). As for populations, wearing consists of aging and death, needing reinvigoration through immigration to survive. Such processes are also meaningful in the modelling of river flow data (see Lawrance and Kottegodan (1977), Lefebvre and Bensalma (2015) and the References to applied works in Vervaat (1979), the modeling of storage problems, pharmacokinetics describing the accumulation of a chemical product in the human body after several injections and also in epidemiology where shocks (formation of clusters) after-effects are slowly attenuated through the process of recovering. Cumulating abruptly increasing work-loads which are being slowly reduced as tasks are being successively accomplished in parallel is also part of the folklore in Computer Science. Such models therefore account for systems where stress is accumulated through abrupt random shocks each slowly released after the shocks (a kind of dual of growth/collapse models whereby stress slowly and continuously accumulates, being interrupted by abrupt release). They are generically non-Markov piecewise deterministic models, however with some notable Markovian exceptions. As is apparent, the key specificity of decay/surge shot-noise models is based on the collective superposition of a random number of declining sub-populations initiated by random shots occurring at Poissonian times. We stress that, even though most decay/surge shot-noise processes are non-Markovian, their embedded chain is, which is a nice feature for simulation purposes.

We start describing the main statistical features of the classical linear shot-noise process with an ad hoc attenuation function (Section 2) before switching to a nonlinear version of it (Section 3) when the attenuation function stems from the flow of some nonlinear dynamical system. We end in (Section 4) with a related class of Markovian shot-noise models akin to (sub-)critical branching processes with immigration, with emphasis on the binary branching process. In the three cases, important questions related to local extinction of the shot-noise process arise, whenever the constituting overlapping subpopulations are allowed to go extinct in finite time.

2. The linear Shot-Noise

We first attempt to describe here the many facets of the linear shot-noise model.

2.1. Generalities

Let Y_n, $n \geq 1$ be a sequence of independent and identically distributed (iid) positive random variables that will represent the amplitudes of the shocks faced by some system in the course of its lifetime. We let $F_Y(dy) = \mathbf{P}(Y \in dy)$, $F_Y(y) = \mathbf{P}(Y \leq y)$, $\overline{F}_Y(y) = \mathbf{P}(Y > y)$ and $\phi_Y(q) = \mathbf{E}(e^{-qY})$, $q \geq 0$ denote the law and Laplace-Stieltjes transform (LST) of Y. We assume that Y has no atom at 0 and

$$(LM): \ \mathbf{E}\left(\log_+ Y\right) < \infty, \tag{1}$$

without much loss of generality on Y (finiteness of the first logarithmic moment condition).

2.1.1. **The underlying counting Poisson process for shocks occurrences**

We further assume that shocks occur at random times S_n, $n \geq 1$ ($S_1 < S_2 < ...$) generated by a possibly time-inhomogeneous Poisson process $P_t = \text{Poi}\left(\int_0^t ds \cdot \lambda_s\right)$ on the positive real line, so with rate $\lambda_t \geq 0$ (assumed a continuous function of time). We require $\Lambda_t = \int_0^t ds \cdot \lambda_s < \infty$, for all $t \in [0, \infty)$. With $P_t = \#\{n : S_n \leq t\}$ therefore, with $s, t > 0$

$$\mathbf{P}\left(P_{t+s} - P_t = n\right) = \frac{(\Lambda_{t+s} - \Lambda_t)^n}{n!} e^{-(\Lambda_{t+s} - \Lambda_t)}.$$

So, $0 = S_0 < S_1 < ... < S_n < ...$ are points of a Poisson process with intensity Λ_t on the real half-line. This means

$$S_n = \Lambda^{-1}\left(S_n^*\right),$$

where $S_1^* < ... < S_n^* < ...$ are points of a rate 1 homogeneous Poisson process and Λ^{-1} is the increasing generalized inverse function of $t \geq 0 \to \Lambda_t$. Consequently, with $n \geq 1$,

$$\mathbf{P}\left(S_n \leq t\right) = \mathbf{P}\left(S_n^* \leq \Lambda_t\right)$$

where $\mathbf{P}\left(S_n^* \leq t\right) = \frac{1}{(n-1)!} \int_0^t e^{-s} s^{n-1} ds$, an Erlang distribution. Denoting $T_n\left(S_{n-1}\right) := S_n - S_{n-1}$, $n \geq 1$, the times elapsed between consecutive jumps of P_t, with distribution $\mathbf{P}\left(T_n\left(S_{n-1}\right) > t\right) = e^{-\int_{S_{n-1}}^{S_{n-1}+t} \lambda_s ds}$, $t \geq 0$, and mean $\mathbf{E}\left(T_n\left(S_{n-1}\right)\right) = e^{\Lambda_{S_{n-1}}} \int_{S_{n-1}}^{\infty} e^{-\Lambda_s} ds$. Note that $\mathbf{P}\left(T_1 > t\right) = \mathbf{P}\left(T_1\left(S_0 = 0\right) > t\right) = e^{-\int_0^t \lambda_s ds}$ is the distribution of the time elapsed before the first jump occurs, and, for $n > 1$

$$\mathbf{P}\left(T_n > t\right) = \int_0^{\infty} \mathbf{P}\left(T_n\left(s\right) > t\right) \mathbf{P}\left(S_{n-1} \in ds\right) =$$

$$\frac{1}{(n-1)!} \int_0^{\infty} ds \lambda_s e^{-(\Lambda_{t+s} - \Lambda_s)} e^{-\Lambda_s} \Lambda_s^{n-1} = \frac{1}{(n-1)!} \int_0^{\infty} d\Lambda_s e^{-\Lambda_{t+s}} \Lambda_s^{n-1}.$$

We stress from this analysis that the sequence (S_n) has the structure of a homogeneous Markov chain which reduces to a sequence with independent increments if the rate λ_t reduces to a constant (when P_t is a homogeneous Poisson process).

We shall limit ourselves to the illustrative case

$$\lambda_t = \lambda t^{a-1} \text{ with } \lambda, a > 0, \tag{2}$$

so that $a = 1$ is the time-homogeneous Poisson case at rate λ.

When $a > 1$ ($a < 1$), there is a speeding up (down) of the jumps' frequencies with time.

When $a > 2$, we have $S_1 < S_2 < ... < S_\infty$ with $S_\infty < \infty$ and the underlying Poisson counting process P_t explodes in finite time.

This is because the explosion criterion of P_t (as a pure jump process) is (see Kersting

and Klebaner (1995))

$$(E): \sum_{n \geq 1} \frac{1}{\lambda_{P_{S_n}}} = \sum_{n \geq 1} \frac{1}{\lambda_n} < \infty. \tag{3}$$

Remark. *Another worthy choice often appearing in the literature is*

$$\lambda_t = \lambda e^{-d \cdot t} \text{ with } \lambda > 0 \text{ and } d \in \mathbb{R}. \tag{4}$$

In this latter case, when $d < 0$ $(d > 0)$, there is an exponential speeding up (down) of the jumps' frequencies with time, with $d < 0$ corresponding to an explosive situation in view of $\sum_{n \geq 1} \frac{1}{\lambda_n} = \lambda^{-1} \sum_{n \geq 1} e^{dn} < \infty$.

2.1.2. *The response function and the linear shot-noise process*

Let $h(t)$, $t \geq 0$ with $h(0) = 1$ be a causal non-negative non-increasing response function translating the way shocks attenuate as time passes by. We assume $h(t) \to 0$ as $t \to \infty$. We also require

$$\int_0^\infty h(s)\, ds < \infty, \tag{5}$$

else $sh(s) \to 0$ as $s \to \infty$.

We allow response functions $h(t)$ with bounded support $[0, t_0]$ so with $h(t) = 0$ if $t \geq t_0 > 0$. As a generic example, with $b \geq 0$, $h(t) = (1 - t/t_0)_+^b$ where $x_+ = \max(x, 0)$.

With $X_0 = x \geq 0$, consider then the shot-noise process

$$X_t = x + \int_0^t \int_{\mathbb{R}_+} y h(t - s)\, \mu(ds, dy), \tag{6}$$

where $\mu(ds, dy) = \sum_{n \geq 1} \delta_{S_n}(ds)\, \delta_{Y_n}(dy)$ (translating independence of the shots' heights and occurrence times). Note that, with $dN_s = \sum_{n \geq 1} Y_n \delta_{S_n}(ds)$, so with $N_t = \sum_{n \geq 1} Y_n \mathbf{1}(S_n \leq t)$ representing a time-inhomogeneous compound Poisson process with jumps' amplitudes Y,

$$X_t = x + \int_0^t h(t - s)\, dN_s, \tag{7}$$

as a linearly filtered compound Poisson process. Under this form, it is clear that X_t cannot be Markov unless $h(t) = e^{-\mu t}$. We shall let

$$\begin{aligned} \nu(dt, dy) &= \mathbf{P}(S_n \in dt, Y_n \in dy \text{ for some } n \geq 1) \\ &= \lambda_t dt \cdot F_Y(dy). \end{aligned}$$

In the sequel, we shall assume without much loss of generality that $x = 0$.

The linear shot-noise process X_t has two alternative equivalent representations, emphasizing its **superposition** characteristics:

$$(1) \quad X_t = \sum_{n \geq 1} Y_n h(t - S_n) \mathbf{1}(S_n \leq t)$$

$$(2) \quad X_t = \sum_{p=1}^{P_t} Y_p h(t - \mathcal{S}_p(t)).$$

Both show that X_t is the size at t of the whole decay/surge population, summing up all the declining contributions of the sub-families which appeared in the past at jump times (a shot-noise or filtered Poisson process model appearing also in Physics and Queuing theory, Snyder (1991), Parzen (1962), Takács (1962), Ross (2020)). The contributions $Y_p h(t - \mathcal{S}_p(t))$, $p = 1, ..., P_t$, of the P_t families to X_t are stochastically ordered in decreasing sizes.

Remark. *Note that defining the new* $[0,1]-$ *valued process* $U_t = e^{-X_t}$, *from representation (1):*

$$U_t = \prod_{n \geq 1} \left(1 - \mathbf{1}(S_n \leq t)\left(1 - e^{-Y_n h(t - S_n)}\right)\right) \tag{8}$$

is a growth-collapse multiplicative model growing between consecutive jump times and collapsing (shrinking) at new jump times.

For the representation (2), let $S_p(t)$ be iid jump times with common law $\mathbf{P}(S(t) \in ds) = \lambda_s ds / \Lambda_t \mathbf{1}(s \in [0,t])$, so supported by the interval $[0,t]$; there are $P_t = \text{Poi}(\Lambda_t)$ such jump events. Denoting by $\mathcal{S}_p(t)$ the ordered version of $S_p(t)$, $p = 1, ..., P_t$ (so with $\mathcal{S}_1(t) < ... < \mathcal{S}_{P_t}(t)$), the joint density of $\mathcal{S}_p(t)$, $p = 1, ..., P_t$ is:

$$f_t(s_1, ..., s_{P_t}) = P_t! \prod_{p=1}^{P_t} \frac{\lambda_{s_p}}{\Lambda_t} ds_p \cdot \mathbf{1}_{0 < s_1 < ... < s_{P_t} < t}.$$

And therefore, the LST of X_t reads

$$\Phi_t^X(q) := \mathbf{E}\left(e^{-qX_t}\right) = \sum_{p \geq 0} \mathbf{P}(P_t = p) \int_{0 < s_1 < ... < s_p < t} p! \prod_{r=1}^{p} \frac{\lambda_{s_r}}{\Lambda_t} \phi_Y(qh(t - s_r)) ds_r$$

$$= e^{-\int_0^t ds \cdot \lambda_s} \left(1 + \sum_{p \geq 1} \int_{0 < s_1 < ... < s_p < t} \prod_{r=1}^{p} \lambda_{s_r} \phi_Y(qh(t - s_r)) ds_r\right).$$

Whence the two expressions of $\Phi_t^X(q) = \mathbf{E}\left(e^{-qX_t}\right) = \mathbf{E}(U_t^q)$:

$$\Phi_t^X(q) = \exp\left\{-\int_0^t ds \cdot \lambda_s \left(1 - \phi_Y(qh(t - s))\right)\right\} \tag{9}$$

$$\Phi_t^X (q) = \exp \left\{ -\Lambda_t \left(1 - \int_0^t ds \cdot \frac{\lambda_s}{\Lambda_t} \phi_Y (qh(t-s)) \right) \right\}. \tag{10}$$

- The first expression as from (9) is known as Campbell's formula, see Campbell (1909a), Campbell (1909b) and Kingman (1992).

- The last equivalent expression of $\Phi_t^X (q)$ in (10) shows that, for each $t > 0$, ($\overset{d}{=}$ meaning equality in distribution)

$$X_t \overset{d}{=} \sum_{p=1}^{P_t} C_p(t), \tag{11}$$

where $C_p(t)$ are iid copies of $C(t)$, the 'typical' clone size at t. The common law of the $C_p(t)$s is characterized by its LST

$$\mathbf{E}\left(e^{-qC_t}\right) = \frac{1}{\Lambda_t} \int_0^t ds \cdot \lambda_s \phi_Y (qh(t-s)), \tag{12}$$

a probability mixture of the LSTs $\phi_Y (qh(s))$ and therefore a LST.

- Concerning the two-point joint distributions, with $q_1, q_2 \geq 0$, $t_2 > t_1$, we have (see Parzen (1962), Theorem $5A$, page 146)

$$\Phi_{t_1,t_2}^X (q_1, q_2) := \mathbf{E}\left(e^{-q_1 X_{t_1} - q_2 X_{t_2}}\right) = \tag{13}$$

$$\exp - \left\{ \begin{array}{l} \int_0^{t_1} ds \cdot \lambda_s \left(1 - \phi_Y (q_1 h(t_1 - s) + q_2 h(t_2 - s))\right) + \\ \int_{t_1}^{t_2} ds \cdot \lambda_s \left(1 - \phi_Y (q_2 h(t_2 - s))\right) \end{array} \right\}.$$

If $\mathbf{E}Y^2 < \infty$, it follows from (9) and (13) that

$$\mathbf{E}X_t = \mathbf{E}Y \int_0^t ds \cdot \lambda_s h(t-s); \ \sigma^2(X_t) = \mathbf{E}Y^2 \int_0^t ds \cdot \lambda_s h^2(t-s)$$

$$\mathrm{Cov}(X_{t_1}, X_{t_1+\tau}) = \mathbf{E}Y^2 \int_0^{t_1} ds \cdot \lambda_s h(t_1 - s) h(t_1 + \tau - s), \ \text{where } t_1, \tau > 0.$$

2.1.3. *Miscellaneous properties*

- Note that the joint law of (P_t, X_t) is characterized by

$$\mathbf{E}\left(z_0^{P_t} e^{-qX_t}\right) = \exp\left\{ -\int_0^t ds \cdot \lambda_s \left(1 - z_0 \phi_Y (qh(t-s))\right) \right\}, \tag{14}$$

so that

$$\mathbf{E}\left(e^{-qX_t} \mid P_t = p\right) = \left[\int_0^t ds \cdot \frac{\lambda_s}{\Lambda_t} \phi_Y (qh(t-s)) \right]^p = \left[\mathbf{E}\left(e^{-qC_t}\right)\right]^p, \ p \geq 0.$$

- The number of shocks above level y over time t : With $P_t(y) = \sum_{p=1}^{P_t} \mathbf{1}(Y_p > y)$, we have

$$
\begin{aligned}
\mathbf{E}\left(z^{\sum_{p=1}^{P_t} \mathbf{1}(Y_p > y)}\right) &= e^{-\int_0^t ds \cdot \lambda_s \left(1 - \mathbf{E}\left(z^{\mathbf{1}(Yh(t-s) > y)}\right)\right)} \\
&= e^{-\int_0^t ds \cdot \lambda_s \mathbf{P}(Yh(t-s) > y)(1-z)},
\end{aligned}
$$

a Poisson random variable (rv) with intensity $\int_0^t ds \cdot \lambda_s \mathbf{P}(Yh(t-s) > y)$. In the homogeneous Poisson case, the intensity reduces to

$$
\lambda \int_0^t ds \cdot \mathbf{P}(Yh(s) > y) = \lambda \int_0^t ds \cdot \overline{F}_Y(y/h(s)).
$$

Supposing $m_n := \mathbf{E}Y^n < \infty$,

$$
k_n(t) = m_n \int_0^t ds \lambda_s h(t-s)^n < \infty
$$

is the $n-$th cumulant of X_t, with in particular $k_1(t) = \mathbf{E}(X_t)$ and $k_2(t) = \sigma^2(X_t)$ and, from (9), if $m_n := \mathbf{E}Y^n < \infty$, for all $n \geq 1$,

$$
-\log \Phi_t^X(q) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} k_n(t) q^n.
$$

This occurs when the moment condition holds

$$
(M): \quad \phi_Y(q) < \infty \text{ for all } q > -q_c \text{ for some } q_c > 0.
$$

2.1.4. **Special cases**

We shall consider two particular cases:

- $h(t) = e^{-\mu t}$, $t \geq 0$, $\mu > 0$. Then

$$
X_t = e^{-\mu t} \int_0^t e^{\mu s} dN_s, \tag{15}
$$

so that

$$
dX_t = -\mu X_t dt + dN_t,
$$

showing that X_t is a time-inhomogeneous Markov process driven by N_t, known as the **classical** linear shot-noise. This is clearly the only choice of the response function that makes X_t Markov. In that case,

$$\Phi_t^X(q) \quad : \quad = \mathbf{E} e^{-qX_t} = e^{-\int_0^t ds \lambda_{t-s}(1-\phi_Y(qe^{-\mu s}))}$$
$$= \quad e^{-\frac{1}{\mu}\int_{e^{-\mu t}}^1 du \lambda_{t+\mu^{-1}\log u}\frac{1-\phi_Y(qu)}{u}}, \text{ where } e^{-\mu s} = u.$$

Supposing $m_n := \mathbf{E} Y^n < \infty$, then

$$k_n(t) \quad = \quad m_n \int_0^t ds \lambda_s e^{-n\mu(t-s)}$$
$$= \quad m_n e^{-n\mu t} \int_0^t ds \lambda_s e^{n\mu s}$$

solving

$$\dot{k}_n(t) = -n\mu k_n(t) + m_n \lambda_t.$$

If $\lambda_t = \lambda t^{a-1}$, $\lambda, a > 0$, when $a \neq 1$ and applying L'Hospital rule,

$$\int_0^t ds \lambda_s e^{n\mu s} = \lambda \int_0^t ds \cdot s^{a-1} e^{n\mu s} \sim (n\mu)^{-1} t^{a-1} e^{n\mu t} \text{ as } t \to \infty.$$

so that

$$k_n(t) \sim m_n t^{a-1} \text{ as } t \to \infty.$$

- If $a > 1$, both $k_1(t) = \mathbf{E}(X_t)$ and $k_2(t) = \sigma^2(X_t)$ grow algebraically at rate $a-1$ and a Central Limit Theorem holds ($\overset{d}{\to}$ meaning convergence in distribution)

$$\frac{X_t - k_1(t)}{\sqrt{k_2(t)}} \overset{d}{\to} \mathcal{N}(0,1) \text{ as } t \to \infty. \tag{16}$$

This is because if $m_3 < \infty$, then $k_3(t) < \infty$ with, Parzen (1962),

$$\frac{k_3(t)}{k_2(t)^{3/2}} \sim \frac{m_3 t^{a-1}}{(m_2 t^{a-1})^{3/2}} = \frac{m_3}{m_2^{3/2}} t^{-(a-1)/2} \to 0 \text{ as } t \to \infty.$$

- If $a < 1$, both $k_1(t) = \mathbf{E}(X_t)$ and $k_2(t) = \sigma^2(X_t) \to 0$ as $t \to \infty$ showing that $X_t \to X_\infty = 0$ with probability 1 (almost sure extinction), but not in finite time. Indeed, this extinction time would be $S_\infty < \infty$ only if $a > 2$.

As $q \to \infty$, we have

$$\Phi_t^X(q) \quad = \quad e^{-\int_0^t ds \lambda_{t-s}(1-\phi_Y(qe^{-\mu s}))}$$
$$\sim \quad e^{-\int_0^t ds \lambda_{t-s}} = e^{-\Lambda_t},$$

due to $\phi_Y(q) \to 0$ as $q \to \infty$ (if Y has no atom at 0). Hence $\mathbf{P}(X_t = 0) = e^{-\frac{\lambda}{a} t^a} =$

$\mathbf{P}\left(S_1 > t\right)$, with

$$\int_0^t \mathbf{P}\left(X_s = 0\right) ds = \frac{1}{a}\int_0^{t^a} e^{-\frac{\lambda}{a}\tau}\tau^{1/a-1}d\tau \to \mathbf{E}\left(S_1\right) = \left(\lambda/a\right)^{-1/a}\Gamma\left(1/a+1\right) \text{ as } t \to \infty.$$

As required because $h\left(t\right) > 0$, there is no visit to zero but initially if $x = 0$.
   - If $a = 1$ $\left(\lambda_t = \lambda\right)$,

$$\Phi_t^X\left(q\right) := \mathbf{E}e^{-qX_t} = e^{-\lambda\int_0^t d\tau\left(1-\phi_Y\left(qe^{-\mu\tau}\right)\right)} = e^{-\frac{\lambda}{\mu}\int_{qe^{-\mu t}}^q dq' \frac{1-\phi_Y\left(q'\right)}{q'}} \tag{17}$$

$$\to e^{-\frac{\lambda}{\mu}\int_0^q dq' \frac{1-\phi_Y\left(q'\right)}{q'}} =: \Phi_\infty^X\left(q\right) \text{ as } t \to \infty.$$

so that X_t has a non-degenerate weak limit X_∞ in the self-decomposable (SD) class if and only if $\mathbf{E}\left(\log_+ Y\right) < \infty$, guaranteeing that (applying Fubini), with $\overline{F}_Y\left(y\right) = \mathbf{P}\left(Y > y\right)$, $y \geq 0$,

$$\int_0^q dq' \frac{1-\phi_Y\left(q'\right)}{q'} = \int_0^\infty \frac{dy}{y}\overline{F}_Y\left(y\right)\left(1 - e^{-qy}\right) < \infty. \tag{18}$$

(see Proposition $A.3.2$ in Steutel (2003)). As $x \to 0$, for some slowly varying function L at zero, by Karamata theorem, we have

$$\mathbf{P}\left(X_\infty \leq x\right) \sim x^{\frac{\lambda}{\mu}}L\left(x\right). \tag{19}$$

See Theorem 1 in Iksanov and Jurek (2003). From the expression of $\Phi_\infty^X\left(q\right)$, X_∞ is in the self-decomposable (SD) class, (see Theorem 2.9 in Steutel (2003)). In particular it is infinitely divisible and unimodal (see Theorem 2.17 in Steutel (2003)).

   Under condition (M), letting $\theta = \lambda/\mu$ and $\psi\left(q\right) = \int_0^q dq'\frac{1-\phi_Y\left(q'\right)}{q'}$, a saddle point estimate yields a large$-x$ estimate of the density of X_∞ as

$$f_{X_\infty}\left(x\right) \sim K\frac{e^{xq_0(x)-\theta\psi(q_0(x))}}{\sqrt{-2\pi\theta\psi''\left(q_0\left(x\right)\right)}}, \tag{20}$$

where $q_0\left(x\right)$ is uniquely defined by $\theta\psi'\left(q_0\left(x\right)\right) = x$. Here $K > 0$ is a proper normalization constant.

   Examples. The simplest explicit case is when $\phi_Y\left(q\right) = 1/\left(1+q\right)$ (else $Y \sim \text{Exp}(1)$) so that

$$\Phi_\infty^X\left(q\right) = \left(1+q\right)^{-\theta}$$

the LST of a Gamma$(\theta, 1)$ distributed rv. Condition (M) is satisfied.
   So long as $\mathbf{E}\left(\log_+ Y\right) < \infty$,

$$\Phi_\infty^X\left(q\right) = e^{-\theta\int_0^q dq'\frac{1-\phi_Y\left(q'\right)}{q'}}$$

is a proper LST and the rv Y needs not have finite moments. As an example, suppose $\phi_Y(q) = 1/(1 + q^\alpha)$ with $\alpha \in (0, 1)$ (modelling heavy-tailed large shocks). The Linnik rv Y only has moments of order smaller than α, Lin (2001), and

$$\Phi_\infty^X(q) = (1 + q^\alpha)^{-\theta/\alpha}.$$

This is again the LST of a Linnik rv, because the rv Y is infinitely divisible; see Lin (2001), for example.

The ergodic case $a = 1$ for which $X_t \xrightarrow{d} X_\infty$ separates a case where X_t is transient at 0 $(0 < a < 1)$ from a case where X_t is transient at ∞ $(a > 1)$. When $a > 2$, state ∞ is in addition reached in finite time.

• If P_t is a homogeneous Poisson process with constant rate $\lambda > 0$ and for any $h(t) > 0$ on $[0, \infty)$ satisfying $\int^\infty h < \infty$,

$$\begin{aligned}\Phi_t^X(q) \quad &: \quad = \mathbf{E}e^{-qX_t} = e^{-\lambda \int_0^t d\tau(1 - \phi_Y(qh(\tau)))} \\ &\rightarrow \quad e^{-\lambda \int_0^\infty d\tau(1 - \phi_Y(qh(\tau)))} = \Phi_\infty^X(q) \text{ as } t \rightarrow \infty,\end{aligned}$$

if X_t has a non-degenerate weak limit. It was proved in Iksanov and Jurek (2003), that a shot-noise process with a characteristic triplet (λ, F_Y, h) has the well defined limit $\Phi_\infty^X(q)$ if and only if,

$$\lambda \int_0^\infty \mathbf{E} \min(1, Yh(s))\, ds < \infty.$$

Note that, upon the change of variable $h(\tau) = u \leftarrow \tau$

$$\Phi_\infty^X(q) = e^{\lambda \int_0^1 dh^{-1}(u)(1 - \phi_Y(qu))} = e^{\lambda \int_0^1 dh^{-1}(u)(1 - \phi_{uY}(q))}, \tag{21}$$

where h^{-1}, the generalized inverse of h, is a decreasing function $[0, 1] \rightarrow [0, \infty]$ with $h^{-1}(u) \rightarrow \infty$ as $u \rightarrow 0$ but $uh^{-1}(u) \rightarrow 0$ as $u \rightarrow 0$ resulting from $\tau h(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$ if $\int^\infty h < \infty$. The function h^{-1} is thus the tail function of a Lévy measure integrating $1 \wedge u$ and so $\int_0^1 dh^{-1}(u)(1 - \phi_{uY}(q)) < \infty$. Under the condition $\int^\infty h < \infty$ and $\mathbf{E}(\log_+ Y) < \infty$ therefore, $\Phi_\infty^X(q)$ is the LST of some proper rv X_∞ which is clearly infinitely divisible.

Examples.

(i) To take a counter-example, suppose $h(\tau) = 1/(1 + \tau)$ and $\phi_Y(q) = 1/(1 + q)$. Then $\Phi_\infty^X(q) = 0$ entailing $X_\infty = \infty$. No non-trivial $\Phi_\infty^X(q)$ exists because $\int^\infty h = \infty$. The attenuation function h has a long memory of the events occurring in the past and keeps track of them, resulting in $X_t \rightarrow \infty$.

(ii) However, choosing $h(\tau) = 1/(1 + \tau^2)$ yields the non-trivial limit (making use in the integration of an arctan)

$$\Phi_\infty^X(q) = e^{-\lambda q \int_0^\infty \frac{d\tau}{1 + q + \tau^2}} = e^{-\lambda \frac{\pi}{2} \frac{q}{\sqrt{1+q}}}.$$

With $\Psi(q) = -\log \Phi_\infty^X(q)$, we have $\Psi'(q) = \lambda \frac{\pi}{2}(1 + q/2)(1 + q)^{-3/2} > 0$, $\Psi''(q) = -\lambda \frac{\pi}{4}(2 + q/2)(1 + q)^{-5/2} < 0,...$, and $\Psi'(q)$ is completely monotone showing that X_∞

is infinitely divisible in the compound Poisson class.

   (iii) Choosing (one-sided stable) $\phi_Y(q) = e^{-q^\alpha}$, $\alpha \in (0,1)$ and $h(\tau) = e^{-\mu\tau}$, yields the well-defined limiting LST

$$\Phi_\infty^X(q) = e^{-\frac{\lambda}{\alpha\mu}\int_0^1 \frac{dv}{v}\left(1-e^{-vq^\alpha}\right)}.$$

2.1.5. *Bounded support response function. First time to (local) extinction and time spent in state* 0

If $h(s)$ has a bounded support, $\int_0^\infty h < \infty$ always and

$$\Phi_t^X(q) = e^{-\int_0^{t\wedge t_0} ds\,\lambda_{t-s}(1-\phi_Y(qh(s)))}.$$

If P_t is a homogeneous Poisson process with constant rate λ, as $t \to \infty$

$$\begin{aligned}
\Phi_t^X(q) &= e^{-\lambda\int_0^{t\wedge t_0} ds(1-\phi_Y(qh(s)))} \\
&\to e^{-\lambda\int_0^{t_0} ds(1-\phi_Y(qh(s)))} = \Phi_\infty^X(q).
\end{aligned}$$

The limit law is reached in finite time t_0. Note that X_∞ has an atom at $x=0$ with

$$\mathbf{P}(X_\infty = 0) = \lim_{q\uparrow\infty} \Phi_\infty^X(q) = e^{-\lambda t_0}.$$

   To take a simple example, supposing $\phi_Y(q) = 1/(1+q)$ and $h(s) = (1-s)_+$, then

$$\Phi_\infty^X(q) = e^{-\lambda\int_0^1 ds\left(1-\frac{1}{1+q(1-s)}\right)} = e^{-\lambda\left(1-\frac{1}{q}\log(1+q)\right)},$$

the LST of a compound Poisson rv with cluster size with $k-$moment $1/(k+1)$.

   Let $Z_n = X_{S_n}$ denote the state of X_t at the jump time S_n, standing for the peaks of X_t (see the following Section). Due to the linearity of the linear shot-noise, the trajectories of X_t can be obtained while launching, for each t between S_n and S_{n+1}, a continuous trajectory $Z_n h(t)$ where $h(t)$ has support $[0, t_0]$. As a result, as soon as $T_{n+1} = S_{n+1} - S_n$ exceeds t_0, an extinction event of X_t occurs for each $t \in [t_0, S_{n+1})$. The first time when this event occurs allows one to compute the time to first extinction as follows. With

$$N = \inf(n \geq 1 : T_{n+1} > t_0),$$

the first time to true local extinction (which is also the first return time to 0 from above if $X_0 = 0$) takes the form

$$\tau_{0,0} = S_N + t_0. \tag{22}$$

We have

$$\mathbf{P}(N = n) = \prod_{m=1}^{n-1} \mathbf{P}(T_{m+1} \leq t_0)\, \mathbf{P}(T_{n+1} > t_0), \ n \geq 1,$$

reducing if P_t is a homogeneous $\Lambda-$Poisson process to the geometric distribution

$$\mathbf{P}\left(N=n\right)=\left(1-e^{-\lambda t_0}\right)^{n-1}e^{-\lambda t_0}.$$

With $t>t_0$, we get

$$\mathbf{P}\left(\tau_{0,0}>t\right)=\int_{\Lambda_{t-t_0}}^{\infty}dse^{-s}\sum_{n\geq1}\frac{\mathbf{P}\left(N=n\right)}{(n-1)!}s^{n-1},\tag{23}$$

which, when P_t is a homogeneous Poisson process, simplifies to the exponential distribution with mean $\lambda^{-1}e^{\lambda t_0}$:

$$\mathbf{P}\left(\tau_{0,0}>t\right)=e^{-\lambda t_0}\int_{\lambda(t-t_0)}^{\infty}dse^{-se^{-\lambda t_0}}=e^{-\lambda e^{-\lambda t_0}(t-t_0)},\ t>t_0.$$

This expression is reminiscent of the Eyring-Kramers time to over-cross an energy barrier for a small noise diffusion process in a double-well potential, Eyring (1935) - Kramers (1940) (escape from a metastable state). The larger the upper bound t_0 of the support of h is, the longer the mean time to first extinction, with an exponential dependence on t_0.

When $h\left(\tau\right)$ has a bounded support $[0,t_0]$, the time X_t spends in the void state 0 during the time interval $[0,t]$, is (assuming $X_0=0$)

$$\int_0^t\mathbf{1}\left(X_s=0\right)ds=T_1\wedge t+\sum_{n\geq2}\left(T_n-t_0\right)_+\mathbf{1}\left(S_n\leq t\right)+(t-t_0)\mathbf{1}\left(S_n\leq t-t_0\right).\tag{24}$$

In the ergodic case, as $t\to\infty$, this zero-set obeys,

$$t^{-1}\int_0^t\mathbf{1}\left(X_s=0\right)ds\to\mathbf{P}\left(X_\infty=0\right)=e^{-\lambda t_0}.\tag{25}$$

Remark. *The renewal rv S_{N+1} is also of interest, being the time elapsed between two consecutive surges following a passage to state 0. It corresponds to the length of the excursions associated to the process X_t and constituting its iid building blocks.*

2.2. The embedded chain

We come back to the linear shot-noise model with impulse function h not necessarily with bounded support. Let $Z_n=X_{S_n}$ denote the state of X_t at the jump time S_n, tracking the peaks of X_t. Note $Z_n>0$ because $Y_n>0$. As emphasized earlier, the trajectories of X_t can be obtained by launching for each t between S_n and S_{n+1} a continuous trajectory $Z_nh(t)$, a useful property in view of the simulation of X_t if Z_n has a Markovian structure. Even though indeed, in most cases but the exceptional Malthus case, X_t is non-Markov, it turns out that the couple (Z_n,S_n) has a Markovian

structure which is a remarkable fact. Indeed, we have

$$
\begin{aligned}
Z_{n+1} &= h\left(T_{n+1}\right) Z_n + Y_{n+1} \\
S_{n+1} &= S_n + T_{n+1}
\end{aligned}
$$

showing that (Z_n, S_n) is in general a bivariate Markov chain whose second marginal only (S_n) is itself a Markov chain, as a result of

$$
\mathbf{P}\left(T_{n+1} > t \mid S_n = s\right) = e^{-\int_s^{s+t} \lambda_{s'} ds'}, \; t \geq 0.
$$

The transition matrix of (Z_n, S_n) may be read from

$$
\mathbf{P}\left(Z_{n+1} \leq z', T_{n+1} \leq t \mid Z_n = z, S_n = s\right)
$$

$$
= \int_0^t \mathbf{P}\left(Y_{n+1} \leq z' - h\left(s'\right) z\right) \mathbf{P}\left(T_{n+1} \in ds' \mid S_n = s\right)
$$

$$
= \int_0^t F_Y\left(z' - h\left(s'\right) z\right) \mathbf{P}\left(T_{n+1} \in ds' \mid S_n = s\right),
$$

where $F_Y(y) = \mathbf{P}(Y \leq y)$, $y \geq 0$. In the last integral, the range of the integration is in fact $\left[h^{-1}\left(z'/z\right) \wedge t, t\right]$ taking into account $F_Y(y) = 0$ if $y \leq 0$. Defining

$$
N(z, t) = \sum_{n \geq 1} \mathbf{1}\left(Z_n > z, S_n \leq t\right)
$$

yields the overshoot intensity

$$
\Lambda(z, t) = \sum_{n \geq 1} \mathbf{P}\left(Z_n > z, S_n \leq t\right),
$$

so with random rate, conditional on (Z_n, S_n)

$$
\lambda(z, t) = \lambda_t \sum_{n \geq 1} \overline{F}_Y\left(z - h\left(T_n\right) Z_{n-1}\right) \mathbf{1}\left(S_{n-1} < t \leq S_n\right).
$$

If P_t is a homogeneous Poisson process with constant rate $\lambda > 0$, then (S_n) is a Markov chain with independent increments and so (Z_n) is itself a Markov chain with transition matrix

$$
\begin{aligned}
\mathbf{P}\left(Z_{n+1} \leq z' \mid Z_n = z\right) &= \mathbf{P}\left(Y_{n+1} \leq z' - h\left(T_{n+1}\right) z\right) \\
&= \lambda \int_{h^{-1}(z'/z)}^{\infty} ds' e^{-\lambda s'} F_Y\left(z' - h\left(s'\right) z\right).
\end{aligned}
$$

In that case, $Z_n \xrightarrow{d} Z_\infty$ as $n \to \infty$, where, with $U = h(T)$ a random variable taking

values in $[0, 1]$ having probability distribution function (pdf) $\mathbf{P}\left(U \leq u\right) = e^{-\lambda h^{-1}(u)}$,

$$Z_\infty \stackrel{d}{=} U Z'_\infty + Y \tag{26}$$

where, with $Z'_\infty \stackrel{d}{=} Z_\infty$, in the RHS (U, Z'_∞, Y) are mutually independent. With (Y_n) an iid sequence independent of U, therefore,

$$Z_\infty \stackrel{d}{=} \sum_{n \geq 0} U^n Y_{n+1},$$

so with

$$\Phi_\infty^Z(q) = \prod_{n \geq 0} \mathbf{E}\left(\phi_Y\left(U^n q\right)\right) \tag{27}$$

as an infinite product of LSTs. Because X_t and Z_n are jointly ergodic or not, this is a well-defined non-degenerate LST under the assumption $\mathbf{E}\left(\log_+ Y\right) < \infty$ and $u h^{-1}(u) \to 0$ as $u \to 0$, Vervaat (1979). Note that, with $Z_n^- = X_{S_n^-}$ the position just before the jump occurring at S_n: $Z_n = Z_n^- + Y_n$ where (Z_n^-, Y_n) are mutually independent. We thus get $Z_n^- \stackrel{d}{\to} Z_\infty^-$ as $n \to \infty$ where Z_∞^- obeys $Z_\infty^- \stackrel{d}{=} U\left(Z_\infty^{-\prime} + Y\right)$, so with

$$\Phi_\infty^{Z^-}(q) = \prod_{n \geq 1} \mathbf{E}\left(\phi_Y\left(U^n q\right)\right). \tag{28}$$

The Markov case. With (Z_∞^-, Y, U) mutually independent, we get

$$\begin{aligned}
\Phi_\infty^{Z^-}(q) &= \int_0^1 F_U\left(du\right) \mathbf{E}\left(e^{-qu(Z_\infty^- + Y)}\right) = \int_0^1 du\, f_U\left(u\right) \Phi_\infty^{Z^-}\left(qu\right) \phi_Y\left(qu\right) \\
&= q^{-1} \int_0^q dq'\, f_U\left(q'/q\right) \Phi_\infty^{Z^-}\left(q'\right) \phi_Y\left(q'\right).
\end{aligned}$$

If, as in the Markov case, $U = h\left(T\right) = e^{-\mu T}$, with $\theta = \frac{\lambda}{\mu} > 0$, $f_U\left(u\right) = \theta u^{\theta-1}$ we get the functional equation

$$\Phi_\infty^{Z^-}(q) = \theta q^{-\theta} \int_0^q dq' \left(q'\right)^{\theta-1} \Phi_\infty^{Z^-}\left(q'\right) \phi_Y\left(q'\right).$$

This can be integrated, leading to

$$\Phi_\infty^{Z^-}(q) = e^{-\theta \int_0^q dq' \frac{1-\phi_Y(q')}{q'}}, \tag{29}$$

with $1 - \phi_Y\left(q'\right) = -\log \psi_Y\left(q'\right)$ for some compound Poisson LST $\psi_Y\left(q'\right)$. Then Z_∞^- is SD with $Z_\infty^- \stackrel{d}{=} X_\infty$. Finally,

$$\Phi_\infty^Z(q) = \phi_Y\left(q\right) \Phi_\infty^{Z^-}\left(q\right) = \phi_Y\left(q\right) e^{-\theta \int_0^q dq' \frac{1-\phi_Y(q')}{q'}}.$$

2.3. Time to failure

Introduce the time-to-failure rv T_c by $\mathbf{P}\left(T_c \in dt \mid X_t\right) = \beta X_t dt$, $\beta > 0$ or equivalently

$$\mathbf{P}\left(T_c > t\right) = \mathbf{E}e^{-\beta \int_0^t X_s ds}. \tag{30}$$

This states that the occurrence rate of a lethal trauma is a linearly increasing function of the residual system stress X_t (which can be the total size or pressure of some population subject to random brutal immigration events each balanced by smooth attenuation aftereffects). This is also particularly meaningful when X_t represents the current total work-load to be achieved. A large level of stress can lead to failure. Then, letting $\overline{X}_t = \int_0^t X_s ds$ and observing

$$\overline{X}_t = \sum_{n \geq 1} Y_n H\left(t - S_n\right) \mathbf{1}\left(S_n \leq t\right),$$

where $H\left(t\right) = \int_0^t h\left(s\right) ds$, by Campbell's formula again,

$$\mathbf{P}\left(T_c > t\right) = e^{-\int_0^t \lambda_s \left(1 - \phi_Y \left(\beta H(t-s)\right)\right) ds}. \tag{31}$$

When Y has all its moments finite, a $\beta-$expansion of the cumulative hazard function $-\log \mathbf{P}\left(T_c > t\right) = -\log \mathbf{E}e^{-\beta \overline{X}_t}$ is available. Supposing $m_n := \mathbf{E}Y^n < \infty$,

$$\overline{k}_n\left(t\right) = m_n \int_0^t ds \lambda_s H\left(t - s\right)^n < \infty$$

is the $n-$th cumulant of $\overline{X}_t$, with in particular $\overline{k}_1\left(t\right) = \mathbf{E}\left(\overline{X}_t\right)$ and $\overline{k}_2\left(t\right) = \sigma^2\left(\overline{X}_t\right)$. From (31)

$$-\log \mathbf{P}\left(T_c > t\right) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \overline{k}_n\left(t\right) \beta^n.$$

• In the homogeneous Poisson(λt) case,

$$\mathbf{P}\left(T_c > t\right) = e^{-\lambda \int_0^t \left(1 - \phi_Y \left(\beta H(s)\right)\right) ds}$$

and

$$\overline{k}_n\left(t\right) = \lambda m_n \int_0^t ds H\left(s\right)^n.$$

When $h\left(s\right) = 1/\left(1 + s^2\right)$, $H\left(s\right) = \arctan\left(s\right)$ and

$$\overline{k}_n\left(t\right) = \lambda m_n \int_0^t ds \arctan\left(s\right)^n = \lambda m_n \int_0^{\arctan t} \frac{u^n}{\cos^2 u} du.$$

In some few examples, an explicit expression of $\mathbf{P}\left(T_c > t\right)$ itself is available from

(31). If, for example, $\phi_Y(q) = 1/(1+q)$ ($Y \sim \text{Exp}(1)$) and $h(s) = e^{-\mu s}$

$$\mathbf{P}(T_c > t) = \left(1 + \frac{\beta}{\mu}\left(1 - e^{-\mu t}\right)\right)^{\lambda/(\mu+\beta)} e^{-\frac{\lambda\beta}{\mu+\beta}t}.$$

2.4. A related Hawkes process

The integrated process is also useful in the context of Cox processes as doubly stochastic Poisson processes $\mathcal{P}_t$ whose intensity $\overline{X}_t := \int_0^t X_s ds$ is the integrated linear random shot-noise process (a variant of a self-exciting Hawkes process, Hawkes (1971), Møller (2003)), so with marginal distribution given by its probability generating function (pgf), Kolmogorov (1935):

$$\Phi_t^{\mathcal{P}}(z) := \mathbf{E}\left(z^{\mathcal{P}_t}\right) = \mathbf{E}e^{-\overline{X}_t(1-z)} = \Phi_t^{\overline{X}}(1-z),$$

where, by Campbell formula

$$\Phi_t^{\overline{X}}(q) = e^{-\int_0^t ds \cdot \lambda_s (1 - \phi_Y(qH(t-s)))}.$$

The jump times of $\mathcal{P}_t$ are thus $\overline{X}^{-1}(S_n^*)$, where $\overline{X}^{-1}(s) := \inf\left(t > 0 : \overline{X}_t > s\right)$.

If they exist, the falling factorial moments $\mu_n(t) = \mathbf{E}(\mathcal{P}_t)_n$ of $\mathcal{P}_t$ are then given by

$$\mu_n(t) = \mathbf{E}\left(\overline{X}_t^n\right) = (-1)^n \cdot {}^{(n)}\Phi_t^{\overline{X}}(0),$$

where ${}^{(n)}\Phi_t^{\overline{X}}(q)$ is the $n-th$ derivative of $\Phi_t^{\overline{X}}(q)$.

In particular, if $\mathbf{E}Y^2 < \infty$,

$$\mathbf{E}\mathcal{P}_t = \mathbf{E}\overline{X}_t = \mathbf{E}Y \int_0^t ds \cdot \lambda_s H(t-s)$$

$$\sigma^2(\mathcal{P}_t) = \mathbf{E}\overline{X}_t + \sigma^2\left(\overline{X}_t\right) = \mathbf{E}Y \int_0^t ds \cdot \lambda_s H(t-s) + \mathbf{E}Y^2 \int_0^t ds \cdot \lambda_s H^2(t-s),$$

showing that $\mathcal{P}_t$ exhibits over-dispersion.

As for the moments $m_n(t) = \mathbf{E}(\mathcal{P}_t^n)$, they are given by $m_n(t) = {}^{(n)}\Phi_t^{\overline{X}}\left(1 - e^\theta\right)|_{\theta=0}$ where ${}^{(n)}\Phi_t^{\overline{X}}\left(1 - e^\theta\right)$ is the $n-th$ derivative of the moment generating function $\Phi_t^{\overline{X}}\left(1 - e^\theta\right)$ with respect to θ. We also have

$$\mathbf{P}(\mathcal{P}_t = 0) = \Phi_t^{\mathcal{P}}(0) = \Phi_t^{\overline{X}}(1) = \exp\left\{-\int_0^t ds \cdot \lambda_s \left(1 - \phi_Y(H(t-s))\right)\right\}$$

$$\mathbf{P}(\mathcal{P}_t = n) = \frac{1}{n!}\mathbf{E}\left(\overline{X}_t^n e^{-\overline{X}_t}\right) = \frac{(-1)^n}{n!} \cdot {}^{(n)}\Phi_t^{\overline{X}}(1).$$

Note that whenever X_t has a weak limit X_∞, by Strong Law of Large Numbers, as $t \to \infty$

$$t^{-1}\mathcal{P}_t \to X_\infty, \text{ almost surely.}$$

The Poisson process $\mathcal{P}_t$ with intensity $\overline{X}_t$ is a population count model enlarging the scope of the linear shot-noise X_t.

2.5. The extremal shot-noise

The extremal shot-noise process is to shot-noise process what Extreme-Value-Theory is to sums of random variables. Define the extremal shot-noise process X_t^* by:

$$X_t^* = \max_{n \geq 1} Y_n h\left(t - S_n\right) \mathbf{1}\left(S_n \leq t\right). \tag{32}$$

It is the largest of the work-loads currently pending in the system, possibly to be handled in priority. It also tracks the family with largest population size among those contributing to X_t. This process was introduced in the Physics' literature in Eliazar and Klafter (2007 and 2009) and a Poisson-field version of such processes was studied in Smith (1990), Dombry (2012). We sketch some of its issues.

By Campbell formula, in case P_t is a homogeneous Poisson process,

$$\mathbf{P}\left(X_t^* \leq x\right) = \exp -\lambda \int_0^t \overline{F}_Y\left(xh\left(s\right)\right) ds. \tag{33}$$

With $x_c = \inf\left(x > 0 : \int_0^\infty \overline{F}_Y\left(xh\left(s\right)\right) ds < \infty\right)$, a limit law with support $[x_c, \infty]$ exists if and only if for all $\infty > x > x_c \geq 0$, $\int_0^\infty \overline{F}_Y\left(xh\left(s\right)\right) ds < \infty$. Whenever $x_c = \infty$, $X_t^* \to \infty$ almost surely and some scaling may be necessary.

Examples.
(i) Suppose $\overline{F}_Y\left(y\right) = \left(1 + y\right)^{-\alpha}$, $\alpha > 0$ and $h\left(s\right) = \left(1 + s\right)^{-2}$. We need to check conditions under which $\int_0^\infty \overline{F}_Y\left(xh\left(s\right)\right) ds < \infty$. For large s, we get $\overline{F}_Y\left(xh\left(s\right)\right) \sim x^{-\alpha} s^{-2\alpha}$ and $\kappa := \int_0^\infty \overline{F}_Y\left(xh\left(s\right)\right) ds < \infty$ if and only if $\alpha > 1/2$, leading to

$$\mathbf{P}\left(X_\infty^* \leq x\right) = \exp -\lambda \kappa x^{-\alpha},$$

a Fréchet limit law with full support. Here $x_c = 0$.

If $\alpha < 1/2$, $x_c = \infty$. Then, with $a_t \sim t^{1/\alpha - 2} \to \infty$,

$$a_t^{-1} X_t^* \overset{d}{\to} F \text{ as } t \to \infty.$$

where F is in the domain of attraction of a Fréchet distribution.

(ii) Suppose $\overline{F}_Y\left(y\right) = e^{-y}$ and $h\left(s\right) \sim \gamma/\log s$ as $s \to \infty$, $\gamma > 0$. Then $\overline{F}_Y\left(xh\left(s\right)\right) \sim s^{-x/\gamma}$ with $\int_0^\infty \overline{F}_Y\left(xh\left(s\right)\right) ds < \infty$ if and only if $x > x_c = \gamma$. In that case, an X_∞^* exists, the law of which has support $[\gamma, \infty]$. Note that for the chosen h, $\int^\infty h = \infty$ so it is not in the class of attenuation functions h that guarantees that a proper X_∞ exists.

(iii) Suppose $h\left(s\right) = e^{-\mu s}$. Then, for all $x > 0$,

$$\int_0^\infty \overline{F}_Y\left(xe^{-\mu s}\right) ds = \frac{1}{\mu} \int_0^x y^{-1} \overline{F}_Y\left(y\right) dy = \infty.$$

In that case, with $a_t = e^{\mu t} \to \infty$, as $t \to \infty$

$$\int_0^t \overline{F}_Y \left(x a_t e^{-\mu s} \right) ds = \frac{1}{\mu} \int_x^{x a_t} y^{-1} \overline{F}_Y (y) \, dy \to \frac{1}{\mu} \int_x^\infty y^{-1} \overline{F}_Y (y) \, dy$$

showing that

$$\mathbf{P} \left(e^{-\mu t} X_t^* \leq x \right) \to e^{-\frac{\lambda}{\mu} \int_x^\infty y^{-1} \overline{F}_Y (y) dy},$$

a well-defined complementary distribution function whatever the distribution of Y.

Remarks.
(i) *When h has bounded support $[0, t_0]$, if $t > t_0$, for all $x > 0$, as $t \to \infty$,*

$$\mathbf{P} \left(X_t^* \leq x \right) = \exp -\lambda \left(\int_0^{t_0} \overline{F}_Y \left(x h(s) \right) ds + t - t_0 \right) \to 0,$$

so with $x_c = \infty$. Here $X_t^* \to \infty$, almost surely and scaling is necessary.
(ii) *Defining $Z_n^* = X_{S_n}^*$, then (Z_n^*) is a Markov chain generated by*

$$Z_{n+1}^* = \max \left(h \left(T_{n+1} \right) Z_n^*, Y_{n+1} \right), \ Z_1^* = Y_1.$$

At some random instants, Z_n^* coincides with the underlying Y_n. With

$$K_n^* = \inf \left(k \geq 1 : Z_{n+k}^* = Y_{n+k} \mid Z_n^* = Y_n \right), \ \text{then}$$

$$T_{n+1}^* = \sum_{k=1}^{K_n^*} T_{n+k}$$

is the time elapsed between consecutive visits of X_t^* to (Y_n).

3. A non-linear Shot-Noise process

So far the response function at time t to a shot of size y appearing at time 0 had the separable form $y_t(y) = y h(t)$. It is linear in the initial condition, with $y_t(y_1 + y_2) = y_t(y_1) + y_t(y_2)$. We now consider situations where this is no longer the case, as when the response function is given by $y_t(y)$, the flow of some declining population model started at y. In this setup, with $t_1, t_2 > 0$, we rather have $y_{t_1 + t_2}(y) = y_{t_1}(y_{t_2}(y))$. The decay patterns will now be the solution trajectories of the nonlinear differential equation (43) below, the solution of which is assumed unique and decaying to zero, either in finite time or not. To a large extent, the previous study of the linear shot-noise, with its many facets, extends to this nonlinear framework. For lack of space, we shall not develop all its aspects as this is an easy parallel extension of the linear case, which could be redundant.

Before proceeding with the non-linear shot-noise construction, we first describe a large family of decaying flows that can be obtained from a time reversal of population growth models.

3.1. From population growth to decay models

We first discuss several deterministic population growth models of the form

$$\dot{y}_t = \alpha\left(y_t\right), \ y_0 = y, \tag{34}$$

where $\alpha\left(y\right)$ is continuous on $[0, \infty)$, positive on $(0, \infty)$ or even sometimes on $[0, \infty)$. We then show how to switch to population decay models that will be of interest to us here.

3.1.1. Some special classes of growth models

Let $y_t > 0$ denote the size (mass) of some population at time $t \geq 0$, with initially $y := y_0 > 0$. With $\alpha_1, c > 0$, consider the growth dynamics

$$\dot{y}_t = \alpha_1 y_t^c, \ y_0 = y, \tag{35}$$

for some growth field $\alpha\left(y\right) := \alpha_1 y^c$. Note that this $\alpha\left(y\right)$ is increasing with y. Integrating when $c \neq 1$ (the non linear case), we get formally

$$y_t\left(y\right) = \left(y^{1-c} + \alpha_1\left(1-c\right)t\right)^{1/(1-c)}, \ y > 0. \tag{36}$$

Three cases arise:

• $0 < c < 1$: then in view of $1/\left(1-c\right) > 1$, the growth of y_t is algebraic at rate larger than 1.

• $c > 1$: then explosion or blow-up of $y\left(t\right)$ occurs in finite time $t_\infty\left(y\right) = y^{1-c}/\left[\alpha_1\left(c-1\right)\right]$. We get

$$y_t\left(y\right) = y\left(1 - t/t_\infty\left(y\right)\right)^{1/(1-c)},$$

with an algebraic singularity. Whenever a growth process exhibits finite time explosion, we say that state ∞ is accessible.

• $c = 1$: this is a simple special case not treated in (35), strictly speaking. However, expanding the solution (36) in the leading powers of $1 - c$ yields consistently:

$$y_t\left(y\right) = e^{\log\left(y^{1-c} + \alpha_1(1-c)t\right)/(1-c)}$$
$$= e^{\log\left[y^{1-c}\left(1+\alpha_1 y^{c-1}(1-c)t\right)\right]/(1-c)} \sim y e^{(1/(1-c))\alpha_1 y^{c-1}(1-c)t} \sim y e^{\alpha_1 t}. \tag{37}$$

This is the simple Malthus growth model.

Remarks.

(i) One can extend the range of c as follows: if $c = 0$, for all $y > 0$, $y\left(t\right) = y + \alpha_1 t$, a linear growth regime. If $c < 0$, (36) holds for all $y > 0$: because $1/\left(1-c\right) < 1$ the growth of y_t is again algebraic but now at rate smaller than 1. In this case however, $\alpha\left(y\right) = \alpha_1 y^c$ is decreasing with y.

(ii) Another example of a growing population with $\alpha\left(y\right)$ decreasing with y is as

follows: Letting $\alpha(y) = \alpha_0 e^{-y}$ leads to a model with slow logarithmic growth:

$$y_t(y) = \log(e^y + \alpha_0 t) = y + \log\left(1 + \frac{\alpha_0}{e^y} t\right). \tag{38}$$

In general $\alpha(y)$ was assumed continuous on $[0, \infty)$, positive on $(0, \infty)$. Then

$$\int_y^{y_t(y)} \frac{dx}{\alpha(x)} = t.$$

Clearly, $t_\infty(y) := \int_y^\infty \frac{dx}{\alpha(x)}$ is the time needed to reach ∞ starting from some y inside the domain.

If for $y > 0$, $t_\infty(y) := \int_y^\infty \frac{dx}{\alpha(x)} < \infty$, (state ∞ accessible), then

$$y_t(y) = t_\infty^{-1}(t_\infty(y) - t),$$

and in general, if this is not the case,

$$y_t(y) = t^{-1}(t(y) + t),$$

where $t(y) = \int^y \frac{dx}{\alpha(x)}$ is an indeterminate integral.

3.1.2. *Related classes of growth models*

With μ, $c > 0$, consider now the dynamics $(\alpha(y) = \mu(1+y)(\log(1+y))^c)$

$$\dot{y}_t = \mu(1 + y_t)(\log(1 + y_t))^c, \ y_0 = y > 0. \tag{39}$$

Introducing $z_t = \log(1 + y_t)$ and $z = \log(1 + y)$, z_t obeys (35) with initial condition z. Integrating (39), we get formally if $c \neq 1$

$$y_t(y) = \exp\left(\left(\log(1+y)\right)^{1-c} + \mu(1-c)t\right)^{1/(1-c)} - 1. \tag{40}$$

We conclude:

- $0 < c < 1$: the integrated solution makes sense and the growth of y_t is exp-algebraic (stretched exponential) at algebraic rate $1/(1 - c) > 1$.

- $c > 1$: an explosion or blow-up of y_t occurs in finite time $t_\infty(y) = (\log(1+y))^{1-c}/[\mu(c-1)]$. We get

$$y_t(y) = (1+y)^{\left(1 - \frac{t}{t_\infty(y)}\right)^{1/(1-c)}} - 1,$$

with an essential singularity.

- $c = 1$: then (39) has a super-exponential solution

$$y_t(y) = (1+y)^{e^{\mu t}} - 1 \text{ for } t \geq 0. \tag{41}$$

Growth occurs at super-exponential (or double exponential) pace. With $t(y) = \int^y \frac{dx}{\mu(1+x)\log(1+x)} = \frac{1}{\mu}\log(\log(1+y))$, one can check that

$$y_t(y) = t^{-1}(t(y) + t).$$

One can extend the range of c as follows: if $c = 0$, $y_t = (1+y)e^{\mu t} - 1$, the Malthusian exponential growth regime. If $c < 0$, from (40) and because $1/(1-c) < 1$, the growth of y_t is exp-algebraic with time, now at algebraic rate smaller than 1.

Let us finally discuss some related choices of $\alpha(y)$.

- $\alpha(y) = \alpha_0 e^y$ leading to

$$y_t(y) = -\log(e^{-y} - \alpha_0 t) = y - \log(1 - t/t_\infty(y)),\ t < t_\infty,$$

which explodes logarithmically at $t_\infty(y) = e^{-y}/\alpha_0$ (a logarithmic singularity).

3.1.3. *Including immigration*

We will now briefly consider two cases involving immigration ($\alpha_0 > 0$).

$$1/\ \alpha(y) = \alpha_0 + \alpha_1 y^c\ \text{(constant immigration rate } \alpha_0\text{)}$$
$$2/\ \alpha(y) = \alpha_0 y + \alpha_1 y^c\ \text{(linear immigration rate } \alpha_0 y\text{)}.$$

Case 1/: The solution to $\dot{y}_t = \alpha(y_t) = \alpha_0 + \alpha_1 y_t^c$, $y_0 = y$ is $y_t(y) = t^{-1}(t(y) + t)$ where

$$t(y) = \int^y \frac{dx}{\alpha_0 + \alpha_1 x^c} = \frac{x}{\alpha_0} F\left(1, \frac{1}{c}, \frac{1}{c} + 1; -\frac{\alpha_1}{\alpha_0} y^c\right),$$

involving the Gauss hypergeometric function $F(a, b, c; z)$. When $c = 1$,

$$y_t(y) = y e^{\alpha_1 t} + \frac{\alpha_0}{\alpha_1}\left(e^{\alpha_1 t} - 1\right), \tag{42}$$

corresponding to a Malthus growth model enhanced by immigration.

Clearly, $t_\infty(y) < \infty \Leftrightarrow c > 1$ (state ∞ accessible in finite time).

Case 2/: The solution to $\dot{y}_t = \alpha(y_t) = \alpha_0 y_t + \alpha_1 y_t^c$, $y_0 = y$ is explicitly known (Bernoulli ode). It is:

$$y_t(y) = e^{\alpha_0 t}\left(y^{1-c} + \frac{\alpha_1}{\alpha_0}\left(1 - e^{-(1-c)\alpha_0 t}\right)\right)^{1/(1-c)},$$

for all $c \neq 1$. When $c = 1$, $y_t(y) = y e^{(\alpha_0 + \alpha_1)t}$ (Malthus), already discussed.

Clearly, $t_\infty(y) < \infty \Leftrightarrow c > 1$ (state ∞ accessible in finite time given by $t_\infty(y) = \frac{1}{(c-1)\alpha_0}\log\left(1 + \frac{\alpha_0}{\alpha_1} y^{1-c}\right)$).

3.1.4. *From growing to declining populations*

A simple time change allows to switch from growing to declining population models.

If $\alpha(y) \to \overline{\alpha}(y) := -\alpha(y)$ where α is as above non-negative, the population size with dynamics

$$\dot{y}_t = -\alpha(y_t), y_0 = y \tag{43}$$

now shrinks as time passes by, starting from $y > 0$. The flow of such decay models is simply obtained while making the substitution $t \to -t$ in the above expressions of $y_t(y)$ with growth rate $\alpha(y)$.

For instance, if $\alpha(y) = \alpha_1 y^c$, respectively $\alpha(y) = \alpha_0(1+y)\log^c(1+y)$, respectively from (36), (40),

$$
\begin{aligned}
y_t(y) &= \left(y^{1-c} + \alpha_1(c-1)t\right)^{1/(1-c)} \\
y_t(y) &= \exp\left(\left(\log(1+y)\right)^{1-c} + \alpha_0(c-1)t\right)^{1/(1-c)} - 1,
\end{aligned}
$$

are the flows associated to $\overline{\alpha}(y) = -\alpha_1 y^c$ and $\overline{\alpha}(y) = -\alpha_1(1+y)(\log(1+y))^c$. Each such flow now goes extinct in finite time if $c < 1$. If $c > 1$, the first flow has algebraic decay with t whereas the second one decays like a stretched exponential.

The declining population model obtained while reversing time of the Malthus model with immigration (42) is

$$y_t(y) = \left(y + \frac{\alpha_0}{\alpha_1}\right)e^{-\alpha_1 t} - \frac{\alpha_0}{\alpha_1}.$$

It corresponds to an exponentially decaying model enhanced by emigration at constant rate $(\overline{\alpha}(y) = -\alpha_0 - \alpha_1 y)$. It goes extinct in finite time $t_0(y) = \frac{1}{\alpha_1}\log\left(\frac{\alpha_0 + \alpha_1 y}{\alpha_0}\right)$.

For a declining population generated by $\overline{\alpha}(y) = -\alpha(y)$, with $0 \le a < y$, the integral

$$t_a(y) := \int_a^y \frac{dx}{\alpha(x)}$$

is the time for the flow to first hit a starting from y.

If $t_0(y) := \int_0^y \frac{dx}{\alpha(x)} < \infty$, there is finite time extinction of $y_t(y)$ and

$$y_t(y) = t_0^{-1}(t_0(y) - t).$$

In general,

$$y_t(y) = t^{-1}(t(y) + t)$$

where $t(y) := -\int^y \frac{dx}{\alpha(x)}$, as an indeterminate integral.

We now have a quite vast class of decaying population models, ranging from logarithmic, algebraic, stretched exponential or doubly exponential decay, some of which can go extinct in finite time, which is the nonlinear version of $h(t)$ having bounded support. These can be obtained while reversing time in (38), (36), (40) and (41), respectively. They are the main ingredients of this Section.

3.2. The non-linear shot-noise model

Consider the decaying flows (43). Define the non-linear shot-noise process as:

$$X_t = \int_0^t \int_{\mathbb{R}_+} y_{t-s}(y) \, \mu(ds, dy).$$

Equivalently, with (Y_n) an iid sequence of initial conditions:

$$X_t = \sum_{n \geq 1} y_{t-S_n}(Y_n) \, \mathbf{1}(S_n \leq t),$$

summing up at t all the family sizes (with initial input Y_n) which occurred in the past at immigration events S_n as points of a Poisson process P_t with rate λ_t. After each shot, each family size decays following the flow (43); see Eliazar and Klafter (2007).

Clearly X_t is non-Markov except when $y_t(y) = y e^{-\mu t}$ (the Malthus case with $\alpha(y) = \mu y$): we are back to the previous formulation with an impulse function $h(t) = e^{-\mu t}$.

Then, by Campbell formula:

$$\Phi_t^X(q) = \exp \left\{ -\int_0^t ds \cdot \lambda_s \left(1 - \mathbf{E} e^{-q y_{t-s}(Y)} \right) \right\}.$$

Clearly, supposing $m_n := \mathbf{E} Y^n < \infty$, in view of $y_t(Y) < Y$, then $\mathbf{E}(y_t(Y)^n) < \infty$ for all $\infty > t \geq 0$. Then

$$k_n(t) = \int_0^t ds \lambda_s \mathbf{E}(y_{t-s}(Y))^n < \infty$$

is the $n-$th cumulant of X_t, with in particular $k_1(t) = \mathbf{E}(X_t)$ and $k_2(t) = \sigma^2(X_t)$. From (9)

$$-\log \Phi_t^X(q) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} k_n(t) q^n.$$

If P_t is homogeneous Poisson at rate λ, under the condition

$$\lambda \int_0^\infty \mathbf{E} \min(1, y_s(Y)) \, ds < \infty,$$

then

$$\Phi_\infty^X(q) = \exp \left\{ -\lambda \int_0^\infty ds \cdot \left(1 - \mathbf{E} e^{-q y_s(Y)} \right) \right\}$$

is the LST of the equilibrium population size.

If for all $n \geq 1$, $k_n = \lambda \int_0^\infty ds \mathbf{E}(y_s(Y))^n < \infty$, then $\Phi_\infty^X(q)$ exists with

$$-\log \Phi_\infty^X(q) = \lambda \int_0^\infty ds \cdot \left(1 - \mathbf{E} e^{-q y_s(Y)} \right) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} k_n q^n.$$

Example:
  - Suppose $\alpha(y) = \alpha_1 y^c$, with $2 > c > 1$. Then,

$$y_s(Y) = \left(Y^{1-c} + \alpha_1(c-1)s\right)^{1/(1-c)} < Y,$$

where $c > 1$ guarantees that there is no finite-time extinction. Integrability of $\mathbf{E}(y_s(Y))^n$ (existence of k_n) is guaranteed if $s^{n/(1-c)}$ is integrable at $s = \infty$, regardless of the initial condition Y. And this is the case for all $n > c - 1$, so for all $n \geq 1$ if $c < 2$.

  The key point here is that $s y_s(Y) \to 0$ as $s \to \infty$, paralleling (5).

  - If $c < 1$, there is extinction of $y_s(Y)$ at the finite time $t_0(Y) = Y^{1-c}/[\alpha_1(1-c)]$, and $\mathbf{E}(y_s(Y))^n$ is always integrable.

More generally, whenever the flow $y_t(y)$ hits 0 in finite time $t_0 = t_0(y)$ (now depending on the initial condition y),

$$\Phi_t^X(q) = \exp\left\{-\lambda \mathbf{E}\int_0^{t \wedge t_0(Y)} ds\left(1 - e^{-q y_s(Y)}\right)\right\},$$

so that $\mathbf{P}(X_t = 0) = e^{-\lambda t \wedge \mathbf{E} t_0(Y)}$. If $\mathbf{E}t_0(Y) < \infty$, as $t \to \infty$:

$$\Phi_t^X(q) \to \exp\left\{-\lambda \mathbf{E}\int_0^{t_0(Y)} ds\left(1 - e^{-q y_s(Y)}\right)\right\},$$

which is always the well-defined infinitely-divisible limiting LST of X_∞. It has an atom at 0: $\mathbf{P}(X_\infty = 0) = e^{-\lambda \mathbf{E} t_0(Y)}$.

  Whenever (S_n) is a standard Poisson sequence and $\mathbf{E}t_0(Y) < \infty$, an estimate of the distribution of the first return time to zero of X_t can be obtained as follows: let X_t^+ denote the shifted nonlinear shot-noise process X_t started at its first jump, so $X_t^+ = X_{t+T_1}$ and let $\tau_+ = \inf\left(t > 0 : X_t^+ = 0\right)$. Recalling $X_0 = 0$, we have

$$\Phi_t^{X^+}(0) := \mathbf{P}\left(X_t^+ = 0\right) = \int_0^t \mathbf{P}\left(\tau_+ \in ds\right)\mathbf{P}\left(X_{t-s} = 0\right).$$

Now,

$$
\begin{aligned}
\Phi_t^X(0) \quad &: \;= \mathbf{P}(X_t = 0) = e^{-\lambda t} + \lambda\int_0^t ds\, e^{-\lambda s}\mathbf{P}\left(X_{t-s}^+ = 0\right) \\
&= \; e^{-\lambda t} + \lambda e^{-\lambda t}\int_0^t ds\, e^{\lambda s}\mathbf{P}\left(X_s^+ = 0\right).
\end{aligned}
$$

Taking the temporal Laplace transforms, with $\Psi^X(p) := \int_0^\infty dt\, e^{-pt}\Phi_t^X(0)$, $\Psi^{X^+}(p) := \int_0^\infty dt\, e^{-pt}\Phi_t^{X^+}(0)$ and $\widehat{\tau}_+(p) := \int_0^\infty e^{-pt}\mathbf{P}(\tau_+ \in dt)$, we get

$$\Psi^X(p) = \frac{1}{\lambda + p}\left(1 + \lambda\Psi^{X^+}(p)\right) \text{ and } \Psi^{X^+}(p) = \widehat{\tau}_+(p)\Psi^X(p).$$

The temporal Laplace transform $\Psi^X(p)$ is the Green kernel of X_t at $(0,0)$ with

$\Psi^X(0) = \infty$ if 0 is visited infinitely often. Recalling $\mathbf{P}(X_t = 0) = e^{-\lambda t \wedge \mathbf{E}t_0(Y)}$, one can check that

$$\Psi^X(p) = \frac{1}{p+\lambda}\left(1 - e^{-(\lambda+p)\mathbf{E}t_0(Y)}\right) + \frac{1}{p}e^{-(\lambda+p)\mathbf{E}t_0(Y)} \text{ and } \Psi^{X^+}(p) = \frac{1}{p}e^{-(\lambda+p)\mathbf{E}t_0(Y)},$$

leading to

$$\widehat{\tau}_+(p) = \mathbf{E}e^{-p\tau_+} = \frac{\lambda+p}{\lambda + pe^{(\lambda+p)\mathbf{E}t_0(Y)}},$$

with $\mathbf{P}(\tau_+ < \infty) = \widehat{\tau}_+(0) = 1$. We also get $\mathbf{E}(\tau_+) = \lambda^{-1}\left(e^{\lambda\mathbf{E}t_0(Y)} - 1\right) = \lambda^{-1}(1 - \mathbf{P}(X_\infty = 0))/\mathbf{P}(X_\infty = 0) < \infty$. Defining $\tau_{0,0} = T_1 + \tau_+$, $\mathbf{E}(\tau_{0,0}) = \lambda^{-1}/\mathbf{P}(X_\infty = 0)$.

3.3. The embedded chain

Let $Z_n = X_{S_n}$. We have

$$\begin{aligned} Z_{n+1} &= y_{T_{n+1}}(Z_n) + Y_{n+1} \\ S_{n+1} &= S_n + T_{n+1} \end{aligned}$$

showing that (Z_n, S_n) is in general a bivariate Markov chain whose second marginal only (S_n) is itself a Markov chain.

The transition matrix of (Z_n, S_n) may be read from

$$\mathbf{P}\left(Z_{n+1} \leq z', T_{n+1} \leq t \mid Z_n = z, S_n = s\right)$$

$$= \int_0^t \mathbf{P}\left(Y_{n+1} \leq z' - y_{s'}(z)\right)\mathbf{P}\left(T_{n+1} \in ds' \mid S_n = s\right)$$

$$= \int_0^t F_Y\left(z' - y_{s'}(z)\right)\mathbf{P}\left(T_{n+1} \in ds' \mid S_n = s\right).$$

If P_t is homogeneous Poisson at rate λ, then (S_n) is a Markov chain with independent increments and so (Z_n) is itself a Markov chain with transition matrix

$$\begin{aligned} \mathbf{P}\left(Z_{n+1} \leq z' \mid Z_n = z\right) &= P\left(Y_{n+1} \leq z' - y_{T_{n+1}}(z)\right) \\ &= \lambda\int_0^\infty ds' e^{-\lambda s'} F_Y\left(z' - y_{s'}(z)\right). \end{aligned}$$

Then if $Z_n \overset{d}{\to} Z$ as $n \to \infty$, then, with $y_T(Z')$ a random variable taking values in $[0, Z']$,

$$Z \overset{d}{=} y_T\left(Z'\right) + Y.$$

Here, $Z' \overset{d}{=} Z$ and in the RHS (T, Z', Y) are mutually independent. This is a non-linear fixed-point equation, with $Z \to y_T(Z)$ a non-linear contracting operator which has or

not solutions.

4. Shot-noise vs branching processes with immigration

In this Section we finally describe a related class of decay/surge process which is Markovian, namely (sub-)critical branching processes with immigration. The specificity of such models is that (i) the decaying dynamics of overlapping subpopulations is now a (sub-)critical branching, so random, process, and (ii) they go extinct in finite time.

A very popular model for quantitatively understanding the emergence of virus resistance both in bacterial colonies and in malignant tumors was introduced in 1943 by Luria and Delbrück (1943). In this setup, individual resistant mutants emerge randomly at birth events embedded in an exponentially growing sensitive bacterial population. The Luria-Delbrück experiment (known as the Fluctuation Test) demonstrates that genetic mutations of bacteria arise permanently, even in the absence of selection, rather than being a response to selection, thereby justifying the latter scenario. It was thus confirmed that mutations do not occur out of necessity (a Lamarckian approach), but instead can occur many generations before the selection strikes (the Darwinian point of view).

It is tacitly assumed in such Luria-Delbrück inspired models that the sensitive population is immune as soon as coexists some mutant in the population; see Huillet (2017). The understanding of the fraction of time spent in a mutant-free state appears desirable because at those local extinction moments, immunity of the sensitive is lost. The Luria-Delbrück model (1943) thus deals with an intertwining of a two-types process (sensitive versus mutant cells), whereby individual resistant mutants collectively emerge (and grow) at random birth events embedded in an exponentially growing sensitive population. The sensitive population grows deterministically at rate $\lambda_t > 0$ and then, on top of it, mutants appear randomly, at a rate proportional to the sensitive population growth rate. In the Lea-Coulson model (1949), mutants arrive one at a time (the surge events) and each mutant subpopulation typically grows, upon appearance, according to a supercritical pure-birth Yule branching process, (see Keller and Antal (2015)). It extends the Luria-Delbrück model where each mutant subpopulation grows deterministically in an exponential way. The Lea-Coulson model is thus a growth-surge type of models appearing in the bacterial resistance to virus; see Kendall (1952) for a survey. A similar point of view arises in carcinogenesis, where $\Lambda_t = \int_0^t ds \cdot \lambda_s$ describes the size at t of the main tumor and side-metastases (with growth rate λ_t driven by the one of the main tumor) play the role of mutants, Kendall (1960), Antal and Krapivsky (2011) and Durrett (2015).

Here we will focus on a large number of incoming mutants variant of the latter model, with three main changes. Firstly, the sensitive population will not necessarily be assumed to grow exponentially. Secondly, mutants will be allowed to arrive at birth events many at a time (and not just one by one); third, instead of growing along a supercritical pure-birth Yule process, the size of each mutant subfamily will now evolve according to a (sub-)critical branching process, including the pure death one. (Sub-)critical branching processes go extinct with probability 1, so questions pertaining to the time to first extinction and time spent in state 0 for the cumulated process arise. In such a way, we are left with a type of decay/surge shot-noise population model with a zero-set. Let us now formulate our model.

4.1. The model

We start with the one-mutant at a time case. Let then a population of wild-type cells grow deterministically at rate $\lambda_t > 0$, with $\Lambda_t = \int_0^t ds \cdot \lambda_s < \infty$, for all $t \in (0,\infty)$. Each wild-type cell is possibly subject to mutation and the rate at which new mutants are being created, one at a time, is $\nu\lambda_t$, where $\nu \in (0,1)$ is the mutation probability of each wild-type cell. A flourishing wild-type population with increasing growth rate with time (such as $\lambda_t = \lambda t^{a-1}$, $a > 1$) favors the adjunction of new mutants accordingly.

The mutant population (growing on top of the wild-type population) is assumed to be resistant to a viral attack in contrast with the wild-type population which is assumed sensitive. Fix a time interval $[0,t]$. Assume mutations occur at iid times $S_p(t)$ with common law $\mathbf{P}(S(t) \in ds) = \lambda_s ds/\Lambda_t$; there are $P(\nu\Lambda_t)$ such mutation events where $P(\nu\Lambda_t) \sim \mathrm{Poi}(\nu\Lambda_t)$ an inhomogeneous Poisson process with intensity $\nu\Lambda_t$. We shall let $P_t := P(\nu\Lambda_t)$.

Once a mutant is created, it grows (or decays) and forms a clone. Let $M_t(1)$ be the mutant sub-population size at t given a unique founder mutant came to birth at time 0. We will assume that $M_t(1)$ evolves according to a Bellman-Harris branching process with $M_0(1) = 1$ and global birth and death rate $r > 0$, the rate at which some branching event occurs. At branching events, each mutant alive is replaced by a random number $M \geq 0$ of offspring, with $\pi_m = \mathbf{P}(M = m)$ and pgf $\phi(z) = \mathbf{E}(z^M) = \sum_{m \geq 0} \pi_m z^m$. We assume that M has finite mean and variance. With $r_b := r\sum_{m \geq 2}(m-1)\pi_m$, $r_s := r\pi_1$ and $r_d := r\pi_0$, respectively the birth, stay alike or death rates, the mutant net death rate is $r_d - r_b = r(1 - \mathbf{E}(M)) = -rf'(1) =: \mu$.

The sub-families pgf $\phi_t(z) := \mathbf{E}(z^{M_t(1)})$ then obeys the Kolmogorov equation

$$\partial_t \phi_t(z) = rf(\phi_t(z)) = rf(z)\partial_z \phi_t(z), \quad \phi_0(z) = z,$$

where $f(z) = \phi(z) - z$ and so, with $\phi_{t_1+t_2}(z) = \phi_{t_1}(\phi_{t_2}(z))$,

$$\int_z^{\phi_t(z)} \frac{dz'}{f(z')} = rt.$$

Whenever $\mu < 0$ ($\mathbf{E}(M) > 1$), respectively $\mu = 0$ ($\mathbf{E}(M) = 1$) and $\mu > 0$ ($\mathbf{E}(M) < 1$), the branching process $M_t(1)$ is super-critical, respectively critical or sub-critical. It accounts for the subsequent growth, stay alike or decline of the mutant subfamilies after their time of appearance. We shall herewith limit ourselves to the (sub)-critical cases $\mu \geq 0$. One can check, observing the branching property $\phi_{t_1,t_2}(z_1,z_2) := \mathbf{E}\left(z_1^{M_{t_1}(1)} z_2^{M_{t_2}(1)}\right) = \phi_{t_1}(z_1\phi_{t_2-t_1}(z_2))$, $t_2 \geq t_1 \geq 0$, that

$$
\begin{aligned}
\mathbf{E}(M_t(1)) &= e^{-\mu t} \ (= 1 \text{ if } \mu = 0) \\
\sigma^2(M_t(1)) &= \left(\frac{\phi''(1)}{\mu} + 1\right)e^{-\mu t}\left(1 - e^{-\mu t}\right) \text{ if } \mu \neq 0 \\
&= r\phi''(1)t \text{ if } \mu = 0 \\
\mathrm{Cov}(M_{t_1}(1), M_{t_2}(1)) &= \left(\frac{\phi''(1)}{\mu} + 1\right)e^{-\mu t_2}\left(1 - e^{-\mu t_1}\right) \text{ if } \mu \neq 0 \\
&= r\phi''(1)t_1 \text{ if } \mu = 0.
\end{aligned}
$$

Whenever $M_t(1)$ is either subcritical or critical ($\mu \geq 0$), $M_t(1)$ goes extinct after a finite extinction time τ_e, with distribution given by

$$\mathbf{P}(M_t(1) > 0) = 1 - \phi_t(0) = \mathbf{P}(\tau_e > t).$$

The time to extinction τ_e is exponentially distributed in the subcritical case, whereas it is Pareto(1) distributed in the critical case, so in any such case, extinction occurs with probability 1. The mutant subpopulations are here assumed to age and end up dying, while on average $\mathbf{E}(M_t(1)) = e^{-\mu t}$ (subcritical) and $\mathbf{E}(M_t(1)) = 1$ (critical).

Let N_t be the size at t of the whole mutant population, summing up all the contributions of the sub-families which appeared in the past at mutation times (here is the shot-noise aspect of this process). With $M_t^{(p)}(1)$ iid copies of $M_t(1)$ and $S_p(t)$ iid copies of $S(t)$, we obtain N_t as a random superposition of birth and death processes initiated at the mutation events

$$N_t = \sum_{p=1}^{P_t} M_{t-S_p(t)}^{(p)}(1). \tag{44}$$

This model clearly also is the one of Markov branching processes with immigration at non-constant rate if $\lambda_s \neq \lambda$, Pakes (1971), Li et al. (2012). Successful (with probability ν) mutation events may be viewed as successful migration events giving rise, at immigration times, to evolving immigrants sub-families along independent branching processes. Therefore an effective mutation event occurs with probability ν whereas an attempted migration event fails with probability $1 - \nu$.

Denoting by $\mathcal{S}_p(t)$ the ordered version of $S_p(t)$, $p = 1, ..., P_t$ (so with $\mathcal{S}_1(t) < ... < \mathcal{S}_{P_t}(t)$), given $P_t = p$, the joint density of $\mathcal{S}_q(t)$, $q = 1, ..., p$ is:

$$f_t(s_1, ..., s_p) = p! \prod_{q=1}^{p} \frac{\lambda_{s_q}}{\Lambda_t} ds_q \cdot 1_{0 < s_1 < ... < s_p < t}.$$

And therefore, the probability generating function (pgf) of N_t reads

$$\begin{aligned}
\Phi_t^N(z) &:= \mathbf{E}\left(z^{N_t}\right) \\
&= \sum_{p \geq 0} \mathbf{P}(P_t = p) \int_{0 < s_1 < ... < s_p < t} p! \prod_{q=1}^{p} \frac{\lambda_{s_q}}{\Lambda_t} \phi_{t-s_q}(z) ds_q \\
&= e^{-\nu \int_0^t ds \cdot \lambda_s} \left(1 + \sum_{p \geq 1} \int_{0 < s_1 < ... < s_p < t} \prod_{q=1}^{p} \lambda_{s_q} \phi_{t-s_q}(z) ds_q\right), \\
&= \exp\left\{-\nu \int_0^t ds \cdot \lambda_s (1 - \phi_{t-s}(z))\right\} \\
&= \exp\left\{-\nu \Lambda_t \left(1 - \int_0^t ds \cdot \frac{\lambda_s}{\Lambda_t} \phi_{t-s}(z)\right)\right\},
\end{aligned} \tag{45}$$

where $\phi_t(z) = \mathbf{E}\left(z^{M_t(1)}\right)$. With $t_2 \geq t_1 \geq 0$, the first expression of $\Phi_t^N(z)$ in (45) has the two-point extension

$$\begin{aligned}
\Phi_{t_1,t_2}^N(z_1, z_2) &= \mathbf{E}\left(z_1^{N_{t_1}} z_2^{N_{t_2}}\right) \\
&= \exp -\nu \left\{\int_0^{t_1} ds \lambda_s (1 - \phi_{t_1-s,t_2-s}(z_1, z_2)) + \int_{t_1}^{t_2} ds \lambda_s (1 - \phi_{t_2-s}(z_2))\right\} \\
&= \exp -\nu \left\{\int_0^{t_1} ds \lambda_s (1 - \phi_{t_1-s}(z_1 \phi_{t_2-t_1}(z_2))) + \int_{t_1}^{t_2} ds \lambda_s (1 - \phi_{t_2-s}(z_2))\right\}.
\end{aligned} \tag{46}$$

One can then check that

$$\mathbf{E}\left(N_t\right) = \int_0^t ds \cdot \lambda_s \cdot \mathbf{E}\left(M_{t-s}\left(1\right)\right) = e^{-\mu t}\int_0^t ds \cdot \lambda_s \cdot e^{\mu s} \ (\ = \Lambda_t \text{ if } \mu = 0)$$

$$\sigma^2\left(N_t\right) = \int_0^t ds \cdot \lambda_s \cdot \mathbf{E}\left(M_{t-s}\left(1\right)^2\right) = \mathbf{E}\left(N_t\right) + \frac{\phi''(1)}{\mu}\int_0^t ds \lambda_{t-s} \cdot e^{-\mu s}\left(1 - e^{-\mu s}\right) \text{ if } \mu \neq 0$$

$$\sigma^2\left(N_t\right) = \Lambda_t + r\phi''\left(1\right)\int_0^t ds \lambda_{t-s}s \text{ if } \mu = 0$$

$$\text{Cov}\left(N_{t_1}, N_{t_2}\right) = \int_0^{t_1} ds \cdot \lambda_s \cdot \mathbf{E}\left(M_{t_1-s}\left(1\right)M_{t_2-s}\left(1\right)\right)$$

$$\text{Cov}\left(N_{t_1}, N_{t_2}\right) = \int_0^{t_1} ds \cdot \lambda_s e^{-\mu(t_2-s)} + \frac{\phi''(1)}{\mu}\int_0^t ds \lambda_s \cdot e^{-\mu(t_2-s)}\left(1 - e^{-\mu(t_1-s)}\right) \text{ if } \mu \neq 0$$

$$\text{Cov}\left(N_{t_1}, N_{t_2}\right) = \Lambda_{t_1} + r\phi''\left(1\right)\int_0^{t_1} ds \cdot \lambda_{t_1-s}s = \sigma^2\left(N_{t_1}\right) \text{ if } \mu = 0.$$

Note the over-dispersion property $\sigma^2\left(N_t\right) > \mathbf{E}\left(N_t\right)$ and

$$\begin{aligned}
&\Phi_t^N\left(0\right) = \mathbf{P}\left(N_t = 0\right) = \exp\left\{-\nu\Lambda_t\left(1 - \int_0^t ds \cdot \frac{\lambda_s}{\Lambda_t}\phi_{t-s}\left(0\right)\right)\right\} \\
&\Phi_{t_1,t_2}^N\left(0,0\right) = \mathbf{P}\left(N_{t_1} = 0, N_{t_2} = 0\right) \\
&= \exp -\nu\left\{\int_0^{t_1} ds \lambda_s\left(1 - \phi_{t_1-s}\left(0\right)\right) + \int_{t_1}^{t_2} ds \lambda_s\left(1 - \phi_{t_2-s}\left(0\right)\right)\right\}.
\end{aligned} \tag{47}$$

The last expression of $\Phi_t^N\left(z\right)$ in (45) shows that, for each t, ($\overset{d}{=}$ meaning equality in distribution)

$$N_t \overset{d}{=} \sum_{p=1}^{P_t} C_t^{(p)}, \tag{48}$$

where $C_t^{(p)}$ are iid copies of C_t, the 'typical' clone size at t. The common law of the $C_t^{(p)}$s is characterized by its pgf

$$\mathbf{E}\left(z^{C_t}\right) = \frac{1}{\Lambda_t}\int_0^t ds \cdot \lambda_s \phi_{t-s}\left(z\right), \tag{49}$$

a probability mixture of the $\phi_s\left(z\right)$s and therefore a pgf. Note that $\mathbf{P}\left(C_t = 0\right) = 0$ if and only if $\phi_t\left(0\right) = \mathbf{P}\left(M_t\left(1\right) = 0\right) = 0$ for all t (the pure birth case). So, if $C_t = 0$ has a positive probability (death is admissible in the binary branching process), N_t is a compound Poisson rv with iid clone sizes possibly zero. This could be adjusted as usual while considering a modified intensity $\nu\Lambda_t \to \nu\Lambda_t\left(1 - \mathbf{P}\left(C_t = 0\right)\right)$ and a new clone size with modified conditional pgf

$$\mathbf{E}\left(z^{C_t}\right) \to \mathbf{E}\left(z^{C_t^+}\right) = \left(\mathbf{E}\left(z^{C_t}\right) - \mathbf{P}\left(C_t = 0\right)\right) / \left(1 - \mathbf{P}\left(C_t = 0\right)\right),$$

where $C_t^+ := C_t \mid C_t > 0$. Several authors study the large$-t$ behavior of C_t^+, Nicholson and Antal (2016).

An equivalent path-wise representation of N_t to the one in (44) is

$$N_t = \sum_{n \geq 1} M_{t-S_n}^{(n)}\left(1\right) \cdot \mathbf{1}_{S_n \leq t}. \tag{50}$$

where $0 = S_0 < S_1 < ... < S_n < ...$ are points of a¨Poisson process with intensity Λ_t on the real half-line.

From these considerations, the expression (45) of the pgf of N_t, namely

$$\Phi_t^N (z) = \exp \left\{ -\nu \int_0^t ds \cdot \lambda_{t-s} (1 - \phi_s (z)) \right\},$$

takes the alternative form

$$\Phi_t^N (z) = \exp \left\{ -\nu \int_z^{\phi_t(z)} dz' \cdot \lambda_{\int_{z'}^{\phi_t(z)} \frac{dy}{rf(y)}} \cdot \frac{1 - z'}{rf(z')} \right\}, \tag{51}$$

while introducing the change of variables $z' = \phi_s (z)$ and while observing $\int_z^{\phi_t(z)} \frac{dy}{rf(y)} = t$ and $\int_z^{z'} \frac{dy}{rf(y)} = s$.

Whenever (S_n) is a standard Poisson sequence with rate λ,

$$\begin{aligned}
\Phi_t^N (z) \quad \rightarrow \quad \Phi_\infty^N (z) &= \exp \left\{ -\nu \lambda \int_0^\infty ds (1 - \phi_s (z)) \right\} \\
&= \exp \left\{ -\frac{\nu \lambda}{r} \int_z^1 dz' \frac{1 - z'}{f(z')} \right\},
\end{aligned}$$

the pgf of a generalized discrete self-decomposable random variable, (see Section 4 pp. 448 in Steutel and van Harn (2003)), so infinitely divisible (else compound Poisson). In that case, the process N_t is Markov and it can be checked that $\Phi_t^N (z)$ solves the PDE:

$$\partial_t \Phi_t^N (z) = rf(z) \partial_z \Phi_t^N (z) + \lambda (z - 1) \Phi_t^N (z), \ \Phi_0^N (z) = 1, \tag{52}$$

involving a local first order (differential) operator, including an additional order zero multiplicative term.

4.2. Related questions and some extensions

4.2.1. Many mutants at a time

So far, $M_t (1)$ represented the size at t of a subpopulation descending from a single ancestor which appeared at time 0. If there are M_0 (possibly random) such ancestors, each of them generating $M_t (1)$ descendants in a mutually independent way, then with $h (z) := \mathbf{E} z^{M_0}$ and $M_t (M_0)$ the size at t of these M_0 ancestors, we have $\mathbf{E} z^{M_t(M_0)} = h \left(\mathbf{E} z^{M_t(1)} \right) = h (\phi_t (1))$. Note that, if M_0 and M_1 are independent with respective pgfs h_0 and h_1 :

$$\mathbf{E} z^{M_t(M_0 + M_1)} = (h_0 h_1) (\phi_t (1)) = \mathbf{E} z^{M_t(M_0)} \mathbf{E} z^{M_t(M_1)},$$

entailing the linearity in law property: $M_t (M_0 + M_1) \overset{d}{=} M_t (M_0) + M_t (M_1)$, where the two terms in the RHS are independent.

Suppose then mutants (or immigrants) when they are being produced enter the system many at a time, describing the amplitudes of the shocks, now random. With

$M_0^{(k)}$ iid positive discrete random variables, independent of $M_{t-S_t^{(k)}}^{(k)}(1)$ and mutually independent, define

$$N_t = \sum_{p=1}^{P_t} M_{t-S_p(t)}^{(p)} \left(M_0^{(p)} \right), \tag{53}$$

where $M_t(M_0) = \sum_{j=1}^{M_0} M_t^{(j)}(1)$ with $M_t^{(j)}(1)$ iid copies of $M_t(1)$. Then

$$\mathbf{E}\left(z^{N_t}\right) = \Phi_t^N(z) = \exp\left\{-\nu \int_0^t ds \cdot \lambda_s \left(1 - h\left(\phi_{t-s}(z)\right)\right)\right\}, \tag{54}$$

where $h(z) = \mathbf{E}\left(z^{M_0}\right)$ is the pgf of M_0, obeying $h(0) = 0$. An equivalent way to see the effect of the mutation probability ν is to introduce the modified pgf

$$\underline{h}(z) := 1 - \nu + \nu h(z),$$

and to consider that the rate of appearance of immigrants is λ_t instead of $\nu\lambda_t$. With probability $\underline{h}(0) = 1 - \nu$ the migration event took place but failed to let any immigrant in, whereas, with probability ν, a random number $M_0 > 0$ of immigrants succeeded entering the system at migration events with rate λ_t. We then let $\underline{h}(z) =: \mathbf{E}\left(z^{\underline{M}_0}\right)$ now for some rv $\underline{M}_0 \geq 0$.

When P_t is homogeneous Poisson at rate λ, the process N_t is a time-homogeneous Markov process and it can be checked that $\Phi_t^N(z)$ solves:

$$\partial_t \Phi_t^N(z) = r f(z) \partial_z \Phi_t^N(z) + \nu\lambda\left(h(z) - 1\right)\Phi_t^N(z), \quad \Phi_0^N(z) = 1 \tag{55}$$

which is a local (differential) operator. In that case

$$\Phi_t^N(z) \quad \rightarrow \quad \Phi_\infty^N(z) = \exp\left\{-\lambda \int_0^\infty ds \left(1 - \underline{h}\left(\phi_s(z)\right)\right)\right\}$$

$$= \quad \exp\left\{-\frac{\lambda}{r}\int_z^1 dz' \frac{1 - \underline{h}(z')}{f(z')}\right\},$$

the pgf of a generalized discrete self-decomposable random variable, (see Section 4 pp. 448 in Steutel and van Harn (2003)). Note that

$$\mathbf{P}\left(N_\infty = 0\right) = \Phi_\infty^N(0) = \exp\left\{-\frac{\lambda}{r}\int_0^1 dz' \frac{1 - \underline{h}(z')}{f(z')}\right\} > 0.$$

Remark. *In the many mutants at a time case, the joint pgf (54) of (N_{t_1}, N_{t_2}), $t_2 \geq t_1 \geq 0$, extends to*

$$\Phi_{t_1,t_2}^N(z_1, z_2) = \exp -\nu \left\{ \begin{array}{l} \int_0^{t_1} ds\lambda_s \left(1 - h\left(\phi_{t_1-s}\left(z_1\phi_{t_2-t_1}(z_2)\right)\right)\right) \\ + \int_{t_1}^{t_2} ds\lambda_s \left(1 - h\left(\phi_{t_2-s}(z_2)\right)\right) \end{array} \right\}.$$

4.2.2. *The joint distribution of $P_t := P(\Lambda_t)$ and N_t*

How many mutation events are responsible for the observation of $N_t = n$ overall mutants at t? This question requires the computation of the joint pgf of (P_t, N_t). It is

$$\mathbf{E}\left(z_0^{P_t} z^{N_t}\right) = \exp\left\{-\nu \int_0^t ds \cdot \lambda_s \left(1 - z_0 h\left(\phi_{t-s}(z)\right)\right)\right\}, \tag{56}$$

so that by Bayes formula

$$\mathbf{P}(P_t = p \mid N_t = n) = \frac{[z_0^p z^n]\, \mathbf{E}\left(z_0^{P_t} z^{N_t}\right)}{[z^n]\, \mathbf{E}\left(z^{N_t}\right)}.$$

Here $[z^n]\,\mathbf{E}\left(z^{N_t}\right)$ denotes the z^n-coefficient of $\mathbf{E}\left(z^{N_t}\right)$.

4.2.3. *The number of mutants with at least m representatives at time t*

Consider the quantity

$$P_t(m) = \sum_{p=1}^{P_t} \mathbf{1}\left(M_{t-S_p(t)}^{(p)}(1) > m\right),$$

counting the number of mutants having appeared in the past with at least $m + 1 \geq 0$ representatives at time t (the frequency spectrum). We have

$$\begin{aligned}
\mathbf{E}\left(z^{P_t(m)}\right) &= \exp\left\{-\nu \int_0^t ds \cdot \lambda_s \left(1 - \mathbf{E}\left(z^{\mathbf{1}\left(M_{t-s}(1) > m\right)}\right)\right)\right\} \\
&= \exp\left\{-\nu(1-z) \int_0^t ds \cdot \lambda_s \mathbf{P}\left(M_{t-s}(1) > m\right)\right\},
\end{aligned}$$

where

$$\mathbf{P}\left(M_t(1) > m\right) = [z^m]\,\frac{1 - \phi_t(z)}{1 - z}.$$

It is a Poisson distributed rv with intensity $\int_0^t ds \cdot \lambda_s \mathbf{P}\left(M_{t-s}(1) > m\right)$. Of particular interest is $P_t(0)$ because $P_t - P_t(0)$ is the number of mutants that appeared in the past, with no current representative at time t. This makes sense only if extinction of subfamilies are possible, requiring $\pi_0 > 0$ in the branching mechanism, so that $\phi_t(0) = \mathbf{P}\left(M_t(1) = 0\right) > 0$.

4.2.4. *A model in the continuum*

Suppose $\mathcal{Y}_t := \sum_{m=1}^{M_t(M_0)} Y_m$ where $(Y_m, m \geq 1)$ is an iid sequence of positive rvs, independent of $M_t(M_0)$. In such situations, not only the number of mutants matters but also the 'charge' Y_m that each mutant carries with it. Put $\phi_Y(q) = \mathbf{E}\left(e^{-qY}\right)$ and

consider

$$X_t = \sum_{p=1}^{P_t} \mathcal{Y}_{t-S_p(t)}^{(p)}.$$

Then, by Campbell formula,

$$\mathbf{E}\left(z_0^{P_t} e^{-qX_t}\right) = \exp\left\{-\nu \int_0^t ds \cdot \lambda_s \left(1 - z_0 h \left(\phi_{t-s} \left(\phi_Y \left(q\right)\right)\right)\right)\right\} \tag{57}$$

is the joint pgf of P_t and Laplace-Stieltjes transform of $X_t > 0$. For such models, at mutation events, a random quantity $\mathcal{Y}$ enters the system and evolves accordingly. Note,

$$\mathbf{E}\left(e^{-qX_t}\right) =: \Phi_t\left(q\right) = \Phi_t^N\left(\phi_Y\left(q\right)\right)$$

and so X_t is Markov as well. The events $X_t = 0$ and $N_t = 0$ coincide.

Whenever (S_n) is a standard Poisson sequence,

$$\Phi_t^X\left(z\right) \to \Phi_\infty^X\left(z\right) = \exp\left\{-\frac{\lambda}{r}\int_{\phi_Y(q)}^1 dz' \frac{1 - h\left(z'\right)}{f\left(z'\right)}\right\} = \Phi_\infty^N\left(\phi_Y\left(q\right)\right), \tag{58}$$

the LST of the limiting random variable X_∞. Note that X_∞ has an atom at 0 with mass

$$\mathbf{P}\left(X_\infty = 0\right) = \Phi_\infty^N\left(0\right) = \exp\left\{-\frac{\lambda}{r}\int_0^1 dz' \frac{1 - h\left(z'\right)}{f\left(z'\right)}\right\},$$

under the additional assumption (together with $\mu > 0$ meaning sub-criticality) $\mathbf{E}\log_+ \underline{M}_0 < \infty$, leading to $\int_0^1 dz' \frac{1 - h(z')}{f(z')} < \infty$.

4.2.5. *The time spent in a mutant-free state: local extinctions*

Let $I_t = \int_0^t \mathbf{1}\left(N_s = 0\right) ds$ be the fraction of the time interval $[0, t]$ free of mutants (the length of the random set uncovered by the mutant sub-populations). We have

$$\mathbf{E}\left(I_t\right) = \int_0^t \mathbf{P}\left(N_s = 0\right) ds = \int_0^t \Phi_s^N\left(0\right) ds \text{ and}$$

$$\mathbf{E}\left[\left(\int_0^t \mathbf{1}\left(N_s = 0\right)\right)^2\right] = \mathbf{E}\left(I_t^2\right) = \int_0^t \int_0^t \mathbf{P}\left(N_{t_1} = 0, N_{t_2} = 0\right) dt_1 dt_2$$

$$= \int_0^t \int_0^t \Phi_{t_1, t_2}^N\left(0, 0\right) dt_1 dt_2.$$

The quantities $\Phi_s^N\left(0\right)$ and $\Phi_{t_1, t_2}^N\left(0, 0\right)$ to be integrated are available from (47).

4.2.6. *An estimate of the first return time to zero*

This question was addressed in Zubkov (1972) but with a different line of proof. Whenever (S_n) is a standard Poisson sequence, an estimate of the distribution of the first return time to zero of N_t can be obtained as follows: let N_t^+ denote the process N_t started at its first successful jump, so $N_t^+ = N_{t+T_1}$ and let $\tau_+ = \inf\left(t > 0 : N_t^+ = 0\right)$. Recalling $N_0 = 0$, we have

$$\Phi_t^{N^+}(0) := \mathbf{P}\left(N_t^+ = 0\right) = \nu \int_0^t \mathbf{P}\left(\tau_+ \in ds\right) \mathbf{P}\left(N_{t-s} = 0\right).$$

Now,

$$\begin{aligned}
\Phi_t^N(0) \quad : \quad &= \mathbf{P}\left(N_t = 0\right) = e^{-\lambda t} + \lambda \int_0^t ds e^{-\lambda s} \mathbf{P}\left(N_{t-s}^+ = 0\right) \\
&= e^{-\lambda t} + \lambda e^{-\lambda t} \int_0^t ds e^{\lambda s} \mathbf{P}\left(N_s^+ = 0\right).
\end{aligned}$$

Taking the Laplace transforms, with $\Psi^X(p) := \int_0^\infty dt e^{-pt} \Phi_t^N(0)$, $\Psi^{X^+}(p) := \int_0^\infty dt e^{-pt} \Phi_t^{N^+}(0)$ and $\widehat{\tau}_+(p) := \int_0^\infty e^{-pt} \mathbf{P}\left(\tau_+ \in dt\right)$, we get

$$\Psi^X(p) = \frac{1}{\lambda + p}\left(1 + \lambda \Psi^{X^+}(p)\right) \text{ and } \Psi^{X^+}(p) = \nu \widehat{\tau}_+(p) \Psi^X(p).$$

So, with $\widehat{\overline{\tau}}_+(p) := \int_0^\infty e^{-pt} \mathbf{P}\left(\tau_+ > t\right) dt = \frac{1}{p}\left(1 - \widehat{\tau}_+(p)\right)$ the LST of the tail distribution of τ_+, and $\widehat{\Psi}(p) := 1 - p\Psi^X(p) = -\int_0^\infty dt e^{-pt} \partial_t \Phi_t^N(0)$, we get

$$\begin{aligned}
\widehat{\tau}_+(p) &= \frac{(\lambda + p)\Psi^X(p) - 1}{\lambda \nu \Psi^X(p)} \text{ and} \\
\widehat{\overline{\tau}}_+(p) &= \frac{1 - p\Psi^X(p)}{\lambda \nu p \Psi^X(p)} = \frac{\widehat{\Psi}(p)}{\lambda \nu \left(1 - \widehat{\Psi}(p)\right)}.
\end{aligned}$$

Recalling $\Phi_0^N(0) = 1$, as $p \to 0$, by the initial value theorem, $\widehat{\Psi}_0(p) \to 1 - \phi$ where $\phi := \Phi_\infty^N(0) \geq 0$, showing that in the subcritical regime, $\widehat{\overline{\tau}}_+(p) \to \mathbf{E}(\tau_+) = (1 - \phi)/(\lambda \nu \phi)$, with $\phi \in (0,1)$. Hence, $\tau_+ < \infty$ with probability 1. The tail distribution is characterized by the pole of $\widehat{\overline{\tau}}_+(p)$, Feller (1957). If the equation $\widehat{\Psi}(p) = 1$ has a real root $-p_0$, with $-\mu \leq -p_0 < 0$, then, as $t \to \infty$, $\mathbf{P}\left(\tau_+ > t\right) \sim \left(\lambda \nu \widehat{\Psi}'(-p_0)\right) e^{-p_0 t}$ (exponential tails) whereas if $\widehat{\Psi}(-\mu) < 1$, then τ_+ has sub-exponential tails. See Zubkov (1972) for more details on this and for the subtleties concerning the critical regime $\mu = 0$. In the critical case indeed, $\Phi_t^N(0) \to \phi := \Phi_\infty^N(0) = 0$ and so $\mathbf{E}(\tau_+) = \infty$.

This gives also the distribution of the first return time to zero from above of X_t defined above. Note that, as $t \to \infty$

$$t^{-1} \int_0^t \mathbf{1}\left(X_s = 0\right) \to \mathbf{P}\left(X_\infty = 0\right).$$

4.2.7. **The embedded chain**

The relation of $N(t)$ with its embedded version is, in part, described in Steutel et al. (1983).

Let $Z_n = N_{S_n}$ (with $Z_0 \overset{d}{=} 0$) be the embedded chain of N_t. Note now $Z_n \geq 0$. We have

$$Z_{n+1} = M_{T_{n+1}}(Z_n) + \underline{M}_0$$

leading, if $\Phi_n^Z(z) = \mathbf{E}\left(z^{Z_n}\right)$, to the recurrence

$$\Phi_{n+1}^Z(z) = \underline{h}(z)\,\Phi_n^Z\left(\underline{\phi}(z)\right),\ \ \Phi_0^Z(z) = 1,$$

the one of a Galton-Watson process with immigration with equivalent branching mechanism

$$\underline{\phi}(z) = \mathbf{E}\left(z^{M_T}\right) = \lambda \int_0^\infty e^{-\lambda t}\phi_t(z)\,dt,$$

and pgf for the number of immigrants $\underline{h}(z)$. We note $\underline{\phi}'(1) = \mathbf{E}(M_T) = \lambda/(\lambda + \mu) < 1$ in the subcritical case $\mu > 0$.

If state 0 is assumed absorbing, let N_t^a be the (absorbed) version of N_t for which $N_t^a = 0$ entails $N_{t+s}^a = 0$ for all $s > 0$. Denoting $Z_n^a = N_{S_n}^a$ the embedded version of N_t^a and $\Phi_n^{Z^a}(z) = \mathbf{E}\left(z^{Z_n^a}\right)$, we get the modified recurrence

$$\Phi_{n+1}^{Z^a}(z) = \underline{h}(z)\,\Phi_n^{Z^a}\left(\underline{\phi}(z)\right) + (1 - \underline{h}(z))\,\Phi_n^{Z^a}(0),\ \ \Phi_0^{Z^a}(z) = 1.$$

Let $\tau_+^{(d)} = \inf\left(n \geq 1 : Z_n^a = 0\right)$ be the discrete time to extinction of Z_n^a (the time to first extinction of Z_n). We have

$$\mathbf{P}\left(\tau_+^{(d)} > n\right) = 1 - \Phi_n^{Z^a}(0) =: \overline{\Phi}_n^{Z^a}(0),$$

obeying the recurrence, $n \geq 0$,

$$\overline{\Phi}_{n+1}^{Z^a}(0) = \underline{h}(0)\,\Phi_n^{Z^a}\left(\underline{\phi}(0)\right) + (1 - \underline{h}(0))\,\overline{\Phi}_n^{Z^a}(0),\ \ \Phi_0^{Z^a}(0) = 1.$$

We can exploit the results of Zubkov (1972) and Seneta and Tavaré (1983) to characterize the tail distribution of the discrete time to first extinction $\tau_+^{(d)}$ of the embedded chain of $N(t)$.

4.3. **The binary branching process example: one mutant at a time**

We illustrate some of the topics just discussed in case of a one mutant at a time (sub-)critical binary branching process with immigration, triggered in most cases by a homogeneous Poisson process P_t.

Whenever $\mathbf{E}(M) < \infty$ and $\sigma^2(M) < \infty$, there is not much loss of generality in assuming that $M_t(1)$ is a binary branching process instead of a general Bellman-Harris one. In such a situation though, M has all its moments finite and the pgf $\phi_t(z)$ has an

explicit form. Upon its appearance here, each mutant duplicates according to a birth (2 offspring with probability π_2), dies according to a death (0 offspring with probability π_0) process, or stays alike with probability π_1, $\pi_0 + \pi_1 + \pi_2 = 1$. We limit ourselves in the analysis below to the case where mutants arrive one at a time ($h(z) = z$). The global birth and death rate is $r > 0$, the rate at which some event, either birth, death or stay alike, occurs. With $r_b := r\pi_2$, $r_s := r\pi_1$ and $r_d := r\pi_0$, then $r = r_b + r_s + r_d$. The mutant net death rate is $\mu := r_d - r_b = r(1 - \mathbf{E}(M)) = r(\pi_0 - \pi_2)$. We assume $\mu \geq 0$ ((sub-)criticality: $\pi_0 \geq \pi_2$) translating the decaying nature of the mutant subpopulations once they have appeared. Each descendant of the original mutant branches in the same way, independently of one another.

In the sequel, we shall let

$$\rho := \pi_0/\pi_2 = r_d/r_b, \ \theta := \lambda/\mu \text{ and}$$
$$\bar{\nu} := \nu\lambda/r_b, \text{ the scaled mutation probability.}$$

For (noncritical) birth and death binary branching processes with $\mu \neq 0$, the sub-families pgf $\phi_t(z) := \mathbf{E}\left(z^{M_t(1)}\right)$ then obeys $\partial_t \phi_t(z) = rf(\phi_t(z))$, $\phi_0(z) = z$, where $f(z) = \pi_0 + \pi_1 z + \pi_2 z^2 - z = (1 - z)(\pi_0 - \pi_2 z)$. The solution is, Harris (1963),

$$\phi_t(z) = \mathbf{E}\left(z^{M_t(1)}\right) = \frac{\pi_0\left(1 - e^{-\mu t}\right) - \left(\pi_2 - \pi_0 e^{-\mu t}\right)z}{\left(\pi_0 - \pi_2 e^{-\mu t}\right) - \pi_2\left(1 - e^{-\mu t}\right)z}$$

$$\frac{1 - \phi_t(z)}{1 - z} = \frac{e^{-\mu t}(\pi_0 - \pi_2)}{\left(\pi_0 - \pi_2 e^{-\mu t}\right) - \pi_2\left(1 - e^{-\mu t}\right)z}.$$

Equivalently, with $\rho = \pi_0/\pi_2 = r_d/r_b$,

$$\phi_t(z) = 1 - \frac{e^{-\mu t}(1 - z)}{1 + \frac{r_b}{\mu}\left(1 - e^{-\mu t}\right)(1 - z)}. \tag{59}$$

In the critical case with $\pi_0 = \pi_2$ ($r_d = r_b$)

$$\phi_t(z) = \mathbf{E}\left(z^{M_t(1)}\right) = 1 - \frac{1 - z}{1 + r_b t(1 - z)}.$$

with $\mathbf{E}(M_t(1)) = 1$ (constant sub-population size on average). We have

$$\mathbf{E}(M_t(1)) = e^{-\mu t} \text{ if } \mu \neq 0, \ \mathbf{E}(M_t(1)) = 1 \text{ if } \mu = 0$$
$$\sigma^2(M_t(1)) = \frac{\pi_0 + \pi_2}{\pi_0 - \pi_2} e^{-\mu t}\left(1 - e^{-\mu t}\right) \text{ if } \mu \neq 0$$
$$= 2r_b t \text{ if } \mu = 0$$

So, with $\mathbf{E}\left(M_t(1)^2\right) = \sigma^2(M_t(1)) + \mathbf{E}(M_t(1))^2$

$$\mathbf{E}\left(M_t(1)^2\right) = \frac{(\pi_0 + \pi_2)e^{-\mu t} - 2\pi_2 e^{-2\mu t}}{\pi_0 - \pi_2} \text{ if } \mu \neq 0$$
$$= 2r_b t + 1 \text{ if } \mu = 0$$

Next, when $\mu \neq 0$,

$$
\begin{aligned}
\mathbf{P}\left(M_t\left(1\right)=0\right) &= \left[z^0\right]\phi_t\left(z\right)=\frac{r_d\left(1-e^{-\mu t}\right)}{r_d-r_be^{-\mu t}} \\
\mathbf{P}\left(M_t\left(1\right)>0\right) &= \frac{\left(r_d-r_b\right)e^{-\mu t}}{r_d-r_be^{-\mu t}} \\
\mathbf{P}\left(M_t\left(1\right)=m\right) &= \left[z^m\right]\phi_t\left(z\right)=\mathbf{P}\left(M_t\left(1\right)>0\right)p_tq_t^{m-1},\ m\geq 1 \\
\mathbf{P}\left(M_t\left(1\right)>m\right) &= \left[z^k\right]\frac{1-\phi_t\left(z\right)}{1-z}=\mathbf{P}\left(M_t\left(1\right)>0\right)q_t^m,\ m\geq 0 \\
\mathbf{P}\left(M_t\left(1\right)>m\mid M_t\left(1\right)>0\right) &= q_t^m,\ m\geq 0.
\end{aligned}
$$

where $q_t = \frac{r_b(1-e^{-\mu t})}{r_d-r_be^{-\mu t}}$, $p_t = 1-q_t$.

$$
\mathbf{P}\left(M_t\left(1\right)>m\right)=\left(r_d-r_b\right)\frac{\left(r_b\left(1-e^{-\mu t}\right)\right)^m}{\left(r_d-r_be^{-\mu t}\right)^{m+1}}.
$$

Note $\mathbf{E}\left(M_t\left(1\right)\right)=\phi_t'\left(1\right)=e^{-\mu t}$.

Depending on $\mu < 0$, $\mu > 0$ or $\mu = 0$, the binary branching process $M_t\left(1\right)$ is super-critical, sub-critical or critical. It accounts for the possibility for each subfamily either to grow or to decline till extinction after its appearance.

Remark. *Although the super-critical case is not our main concern, we mention some of its aspects as it appears in the Luria-Delbrück model of bacterial resistance to virus. In the super-critical case ($\mu < 0$), extinction occurs with positive probability at time τ_e. We have*

$$
1-\phi_t\left(0\right)=\mathbf{P}\left(\tau_e>t\right)=\frac{e^{-\mu t}}{1-\frac{r_b}{\mu}\left(e^{-\mu t}-1\right)} \tag{60}
$$

and

$$
\mathbf{P}\left(\tau_e<\infty\right)=1+\frac{\mu}{r_b}=\frac{r_d}{r_b}=\rho\ \text{if}\ \mu<0.
$$

So, in the super-critical birth and death case, $\rho \in (0,1)$ is the probability of extinction of $M_t\left(1\right)$, as the smallest solution in $[0,1]$ of $f\left(z\right)=0$. And, given extinction has occurred, the tail of the distribution of τ_e is exponential with mean $-1/\mu$.

Note that if $\mu < 0 : q_t \sim 1+\frac{\mu}{r_b}e^{\mu t}$ as $t \to \infty$, so that for all $x > 0$

$$
\mathbf{P}\left(M_t\left(1\right)e^{\mu t}>x\mid M_t\left(1\right)>0\right)=q_t^{xe^{-\mu t}}\sim e^{\frac{\mu}{r_b}x}\ \text{as}\ t\to\infty,
$$

showing that, as $t \to \infty$,

$$
e^{\mu t}M_t\left(1\right)\mid\left(M_t\left(1\right)>0\right)\xrightarrow{d}\mathrm{Exp}\left(-\mu/r_b\right)
$$

an exponential rv with scale parameter $-\mu/r_b$. Because $\mathbf{P}\left(M_t\left(1\right)>0\right)\sim 1-\rho$ we get

$$
M_t\left(1\right)\xrightarrow{d}\rho\delta_0+\left(1-\rho\right)e^{-\mu t}\mathrm{Exp}\left(-\mu/r_b\right)\ \text{as}\ t\to\infty.
$$

There are two extreme cases:

- the Yule-Furry pure birth case is when $\pi_0 = 0$ else $r_d = 0$ and $\mu = -r_b = -r\pi_2 < 0$, Yule (1925). Then

$$\phi_t(z) = 1 - \frac{e^{-\mu t}(1-z)}{1 + (e^{-\mu t} - 1)(1-z)} = \frac{ze^{\mu t}}{1 - z(1 - e^{\mu t})},$$

a geometric pgf with success probability $1 - e^{-r_b t}$. Here

$$e^{\mu t} M_t(1) \xrightarrow{d} \text{Exp}(1), \text{ as } t \to \infty,$$

(exponential growth of $M_t(1)$).

- the Greenwood pure death case is when $\pi_2 = 0$ else $r_b = 0$ and $\mu = r_d = r\pi_0 > 0$. Then

$$\phi_t(z) = 1 - e^{-\mu t}(1-z),$$

a Bernoulli pgf with success probability $e^{-r_d t}$. Here, for all $m > 0$

$$\mathbf{P}(M_t(1) > m) = [z^m] \frac{1 - \phi_t(z)}{1 - z} = e^{-\mu t} \to 0, \text{ as } t \to \infty$$

and $\mathbf{P}(M_t(1) = 0) = 1 - e^{-\mu t} \to 1$, as $t \to \infty$ ($M_t(1) \xrightarrow{d} \delta_0$ with $\text{Exp}(\mu)$ extinction time distribution).

To estimate the large time t behavior of N_t in (51), we need to evaluate

$$\int_{z'}^{\phi_t(z)} \frac{dy}{rf(y)} = \frac{1}{\pi_0 r} \int_{z'}^{\phi_t(z)} \frac{dy}{(1-y)(1-y/\rho)}.$$

Note that, depending on $\rho < 1$ or $\rho \geq 1$, the dominant singularity of $1/f$ is at ρ or at 1. We have

$$\int_{z'}^{\phi_t(z)} \frac{dy}{rf(y)} = \frac{1}{(\pi_2 - \pi_0)r} \left(\log \frac{1 - \phi_t(z)}{\pi_0 - \pi_2 \phi_t(z)} \frac{\pi_0 - \pi_2 z'}{1 - z'} \right).$$

If $\rho > 1$ (subcritical case: $\pi_0 > \pi_2$ or $\mu > 0$), $\phi_t(z) \to 1$ (translating that $M_t(1) \to 0$), so with $1 - \phi_t(z) \sim e^{-\mu t} \frac{\mu(1-z)}{r_d - r_b z}$. Thus

$$\int_{z'}^{\phi_t(z)} \frac{dy}{rf(y)} \sim \frac{1}{(\pi_2 - \pi_0)r} \log(1 - \phi_t(z)) \sim t - \frac{1}{\mu} \log \frac{\mu(1-z)}{r_d - r_b z}.$$

A large t estimate of $\lambda_{\int_{z'}^{\phi_t(z)} \frac{dy}{rf(y)}}$ appearing in (51) is thus

$$\lambda_{\int_{z'}^{\phi_t(z)} \frac{dy}{rf(y)}} \sim \lambda_{t - \frac{1}{\mu} \log \frac{\mu(1-z)}{r_d - r_b z}}.$$

We get

$$
\begin{aligned}
\Phi_t^N(z) \;&\sim\; \exp\left\{ -\frac{\nu}{r}\lambda_{t-\frac{1}{\mu}\log\frac{\mu(1-z)}{r_d-r_b z}} \int_z^{\phi_t(z)} dz' \cdot \frac{1-z'}{f(z')} \right\} \\
&=\; \exp\left\{ -\frac{\nu}{r}\lambda_{t-\frac{1}{\mu}\log\frac{\mu(1-z)}{r_d-r_b z}} \int_z^{\phi_t(z)} dz' \cdot \frac{1}{\pi_0-\pi_2 z'} \right\} \\
&=\; \exp\left\{ \frac{\nu}{\pi_2 r}\lambda_{t-\frac{1}{\mu}\log\frac{\mu(1-z)}{r_d-r_b z}} \log\left(\frac{\pi_0-\pi_2\phi_t(z)}{\pi_0-\pi_2 z} \right) \right\} \\
&\sim\; \exp\left\{ \frac{\nu}{\pi_2 r}\lambda_{t-\frac{1}{\mu}\log\frac{\mu(1-z)}{r_d-r_b z}} \log\left(\frac{\pi_0-\pi_2}{\pi_0-\pi_2 z} \right) \right\} \\
&\sim\; \left(\frac{\pi_0-\pi_2}{\pi_0-\pi_2 z} \right)^{\frac{\nu}{r_b}\lambda_t}
\end{aligned}
$$

• In case of a sub-linear wild-type population growth, $\lambda_t \to 0$ as $t \to \infty$ and so, with $\rho > 1$

$$
\begin{aligned}
\Phi_t^N(z) \;&\sim\; 1 + \frac{\nu}{r_b}\lambda_t \log\left(\frac{\pi_0-\pi_2}{\pi_0-\pi_2 z} \right) \text{ and} \\
[z^0]\,\Phi_t^N(z) \;&=\; \mathbf{P}(N_t=0) \sim 1 + \frac{\nu}{r_b}\lambda_t \log(1-1/\rho) \\
[z^n]\,\Phi_t^N(z) \;&=\; \mathbf{P}(N_t=n) \sim \frac{\nu}{r_b}\lambda_t \frac{1}{n}\rho^{-n},\ n\geq 1,
\end{aligned}
$$

emphasizing that asymptotically, $N_t \to 0$ with probability 1.

• If $\lambda_t \to \infty$ as $t \to \infty$ then,

$$
\mathbf{E}\left(e^{-qN_t/\left(\frac{\nu}{r_b}\lambda_t\right)} \right) \sim \left(\frac{\pi_0-\pi_2}{\pi_0-\pi_2 e^{-q/\left(\frac{\nu}{r_b}\lambda_t\right)}} \right)^{\frac{\nu}{r_b}\lambda_t} \sim e^{-q/(\rho-1)},
$$

showing that $N_t/\left(\frac{\nu}{r_b}\lambda_t\right) \to 1/(\rho-1)$ almost surely.

• In the constant rate case with $\lambda_t = \lambda > 0$ a constant, and in the subcritical case $\mu > 0$

$$
\Phi_t^N(z) = \exp\left\{ -\frac{\nu\lambda}{r} \int_z^{\phi_t(z)} dz' \frac{1-z'}{f(z')} \right\} = \left(\frac{\pi_0-\pi_2\phi_t(z)}{\pi_0-\pi_2 z} \right)^{\frac{\nu\lambda}{r_b}} \to \left(\frac{\pi_0-\pi_2}{\pi_0-\pi_2 z} \right)^{\frac{\nu\lambda}{r_b}},
$$

so that N_t converges in distribution to a negative binomial distribution with mean $\mathbf{E}(N_\infty) = \nu\lambda/\mu$ and variance $\sigma^2(N_\infty) = \nu\lambda r_d/\mu^2$. Only finitely many mutants are present in the population for large time. The pgf of the global mutant population size at t may be written as $\Phi_t^N(z) = \left(1 + \frac{r_b}{\mu}\left(1-e^{-\mu t}\right)(1-z) \right)^{-\overline{\nu}}$.

4.3.1. **The mean of** I_t

Considering $I_t = \int_0^t \mathbf{1}\,(N_s = 0)\,ds$, the fraction of the time interval $[0, t]$ free of mutants, we therefore get

$$
\mathbf{E}\,(I_t) = \int_0^t \Phi_s(0)ds = \int_0^t \left(1 + \frac{r_b}{\mu}\left(1 - e^{-\mu s}\right)\right)^{-\bar\nu} ds
$$

$$
= \frac{1}{\mu}\left(1 + \frac{r_b}{\mu}\right)^{-\bar\nu} \int_{\frac{r_b}{r_d}e^{-\mu t}}^{\frac{r_b}{r_d}} \frac{du}{u}(1-u)^{-\bar\nu} \underset{t\to\infty}{\sim} \left(1 + \frac{r_b}{\mu}\right)^{-\bar\nu} t
$$

(the latter equivalence is justified by the fact that the main contribution of the integral is near $u = 0$:

$$
\int_{\frac{r_b}{r_d}e^{-\mu t}}^{\frac{r_b}{r_d}} \frac{du}{u}(1-u)^{-\bar\nu} \sim \int_{\frac{r_b}{r_d}e^{-\mu t}}^{\frac{r_b}{r_d}} \frac{du}{u} = \log\left(\frac{r_b}{r_d}\right) - \log\left(\frac{r_b}{r_d}e^{-\mu t}\right) \sim \mu t\right).
$$

Hence, if $\mu > 0, \mathbf{E}\,(I_t) \underset{t\to\infty}{\sim} \left(1 + \frac{r_b}{\mu}\right)^{-\bar\nu} t$. With $\left(1 + \frac{r_b}{\mu}\right)^{-\bar\nu} = \left(\frac{\mu}{r_d}\right)^{\bar\nu} \in (0,1)$, a whole constant fraction of the real line is left uncovered by the mutants.

Remark. In this subcritical case, the process $B_t := \mathbf{1}\,(N_t = 0)$ is ergodic and by Strong Law of Large Numbers, consistently,

$$
\frac{1}{t}I_t \overset{\text{a.s.}}{\underset{t\to\infty}{\to}} \mathbf{E}\,(\mathbf{1}\,(N_\infty = 0)) = \mathbf{P}\,(N_\infty = 0) = \left(\frac{\mu}{r_d}\right)^{\bar\nu}.
$$

4.3.2. **The zero-set of** $\{N_t\}$ **as an alternating renewal process in the subcritical case**

In the subcritical case, the system alternates between time periods for which $N_t > 0$ (the covered zones) and those for which $N_t = 0$ (the uncovered zones). We wish to understand some of the features of this alternating renewal process, akin to the one of an $M/G/\infty$ queue in Kendall's notations, see Takács, L. (1962).

Consider the set $\{t \geq 0 : N_t = 0 \text{ and } N_{t+dt} > 0\}$ and assume $t = 0$ belongs to this set, with $N_{dt} = 1$. Let

$$
\phi_t\,(0) = \mathbf{P}\,(\tau_e \leq t) = \frac{\left(1 + \frac{r_b}{\mu}\right)\left(1 - e^{-\mu t}\right)}{1 + \frac{r_b}{\mu}\left(1 - e^{-\mu t}\right)}
$$

be the extinction time of this individual.

With $\mathcal{N}_t := \#\{0 \leq s \leq t : N_t = 0 \text{ and } N_{t+dt} > 0\}$ and $U\,(t) := \mathbf{E}\,(\mathcal{N}_t)$ its renewal function,

$$
U\,(t) = \mathbf{P}\,(N_t = 0) + \nu\lambda \int_0^t ds\mathbf{P}\,(N_s = 0),
$$

where $\mathbf{P}\,(N_t = 0) = e^{-\nu\lambda \int_0^t (1-\phi_s(0))ds}$. $U\,(t)$ is differentiable with renewal rate function

$$
\dot{U}\,(t) =: u\,(t) = -\nu\lambda\,(1 - \phi_t\,(0))\,\mathbf{P}\,(N_t = 0) + \nu\lambda\mathbf{P}\,(N_t = 0) = \nu\lambda\phi_t\,(0)\,\mathbf{P}\,(N_t = 0).
$$

Let T be the random time separating two consecutive points of the renewal set $\mathcal{N}_\infty$ and $\phi_T(p) := \mathbf{E}\left(e^{-pT}\right)$ the Laplace transform of its distribution, $p \geq 0$ (The rv T constitutes the length of the excursions of N_t). Then, by classical renewal arguments in Ross (2010)

$$\phi_T(p) = 1 - \frac{1}{p\widehat{U}(p)} = \frac{\widehat{u}(p)}{1 + \widehat{u}(p)},$$

where $\widehat{U}(p) = \int_0^\infty dt e^{-pt} U(t)$ and $\widehat{u}(p) = \int_0^\infty dt e^{-pt} u(t)$. Letting $A(p) := \int_0^\infty dt e^{-pt} \mathbf{P}(N_t = 0)$, after an integration by parts, we get

$$\widehat{U}(p) = A(p) + \nu\lambda \int_0^\infty dt e^{-pt} \int_0^t ds \mathbf{P}(N_s = 0) = A(p)\left(1 + \frac{\nu\lambda}{p}\right).$$

Therefore

$$\phi_T(p) = 1 - \frac{1}{A(p)(p + \nu\lambda)}.$$

We have $\mathbf{P}(N_t = 0) \underset{t\to\infty}{\to} e^{-\nu\lambda a}$ where $a := \int_0^\infty (1 - \phi_s(0)) ds = \log(1 + r_b/\mu)/r_b$ and by Tauberian theorem $pA(p) \underset{p\to 0}{\to} e^{-\nu\lambda a} = (1 + r_b/\mu)^{-\bar\nu}$. Thus

$$\phi_T(p) \underset{p\to 0}{\sim} 1 - p\frac{e^{\nu\lambda a}}{\nu\lambda},$$

showing that $\mathbf{E}(T) = e^{\nu\lambda a}/(\nu\lambda)$. Clearly, $T = \underline{T} + E$ where $\underline{T}$ is the time interval over which $N_t > 0$ (the covered zones) and E the time interval over which $N_t = 0$ (the uncovered zones). Besides, $\underline{T}$ and E are independent and $E \sim \exp(\nu\lambda)$. Therefore

$$\phi_{\underline{T}}(p) = \frac{p + \nu\lambda}{\nu\lambda}\phi_T(p) = 1 - \frac{1}{\nu\lambda}\left(\frac{1}{A(p)} - p\right),$$

with $\mathbf{E}(\underline{T}) = \left(e^{\nu\lambda a} - 1\right)/(\nu\lambda)$. Let us compute the second moments of T and $\underline{T}$ and check their finiteness. With $B(p) := 1 - pA(p)$, we have $\left(1 - \phi_{\underline{T}}(p)\right)(1 - B(p)) = (\nu\lambda)^{-1} p B(p)$. Differentiating twice with respect to p, using the Leibniz rule and evaluating the result at $p = 0$,

$$\begin{aligned}
\mathbf{E}\left(\underline{T}^2\right) &= \frac{2e^{2\nu\lambda a}}{\nu\lambda} \int_0^\infty \left(\mathbf{P}(N_t = 0) - \mathbf{P}(N_\infty = 0)\right) dt \\
&= \frac{2e^{2\nu\lambda a}}{\nu\lambda} \int_0^\infty \left(e^{-\nu\lambda \int_0^t (1-\phi_s(0))ds} - e^{-\nu\lambda a}\right) dt
\end{aligned}$$

$$= \frac{2e^{\nu\lambda a}}{\nu\lambda} \int_0^\infty \left(e^{\nu\lambda \int_t^\infty (1-\phi_s(0))ds} - 1 \right) dt$$

$$= \frac{2e^{\nu\lambda a}}{\nu\lambda} \int_0^\infty \left(\left(1 - \frac{r_b e^{-\mu t}}{r_d}\right)^{-\bar{\nu}} - 1 \right) dt$$

$$= \frac{2e^{\nu\lambda a}}{\nu\lambda\mu} \sum_{k\geq 1} \frac{[\bar{\nu}]_k}{kk!} \left(\frac{r_b}{r_d}\right)^k < \infty.$$

And

$$\mathbf{E}\left(T^2\right) = \mathbf{E}\left(\underline{T}^2\right) + \frac{2}{\nu\lambda}\mathbf{E}(\underline{T}) + \frac{2}{(\nu\lambda)^2} = \mathbf{E}\left(\underline{T}^2\right) + 2\frac{e^{\nu\lambda a}}{(\nu\lambda)^2} < \infty.$$

We note that in both cases, consistently with (Ross (2010), p. 450),

$$\mathbf{P}\left(N_\infty = 0\right) = \frac{\mathbf{E}(E)}{\mathbf{E}(E) + \mathbf{E}(\underline{T})} = e^{-\nu\lambda a} = (1 + r_b/\mu)^{-\bar{\nu}} = \left(\frac{\mu}{r_d}\right)^{\bar{\nu}}.$$

4.4. The critical case

For critical birth and death processes, with $\phi_t(z) = \mathbf{E}\left(z^{M_t(1)}\right)$,

$$1 - \phi_t(z) = \frac{1-z}{1 + r_b t(1-z)}.$$

Then the clone size pgf reads

$$\mathrm{E}\left(z^{C_t}\right) = \int_0^t ds \cdot \frac{\lambda_s}{\Lambda_t} \phi_{t-s}(z) = \frac{1}{t} \int_0^t ds \cdot \phi_{t-s}(z)$$

$$= 1 - \frac{1-z}{t} \int_0^t ds \cdot \frac{1}{1 + r_b s(1-z)}$$

$$= 1 - \frac{1}{r_b t} \log\left(1 + r_b t(1-z)\right)$$

$$\underset{t\to\infty}{\to} 1 = \mathbb{E}\left(z^{C_\infty}\right), \text{ with } C_t \overset{d}{\to} 0.$$

The global mutant population size at $t$pgf is

$$\Phi_t^N(z) = \exp\left\{-\nu\lambda(1-z)\int_0^t \frac{1}{1 + r_b s(1-z)}ds\right\}$$

$$= (1 + r_b t(1-z))^{-\nu\lambda/r_b},$$

the pgf of a (discrete-self-decomposable) negative binomial distribution with mean $\mathbf{E}\left(N_t\right) = \nu\lambda t$ and variance $\sigma^2\left(N_t\right) = \nu\lambda t\left(1 + r_b t\right) \sim \nu\lambda r_b t^2$. Summing up infinitely many (Poi (rt)) clones of size tending to 0 gives rise to a time-dependent random limit

for N_t. With $\bar{\nu} := \nu\lambda/r_b$, we also have, while considering a Laplace-Stieltjes transform:

$$
\mathbf{E}\left(c^{-\alpha N_t/(\nu\lambda t)}\right) = \left(1 + r_b t \left(1 - e^{-q/(\nu\lambda t)}\right)\right)^{-\bar{\nu}}
$$

$$
\underset{t\to\infty}{\sim} \left(1 + q\frac{r_b}{\nu\lambda}\right)^{-\bar{\nu}},
$$

showing that $N_i/(\nu\lambda t)$ converges in law to a gamma($\nu\lambda/r_\mu, \nu\lambda/r_d$) d stribution with mean 1 .

Remarks. - In the eritial case under study here, N_i no longer converyes to some limiting rv N_∞ as in the subcritical case, exen though here-also the subfamilies go crtinct wxith probability Li ergodicity is brokeri by the lang time it takes in the critical case to reach extinction. - If $r_b \to 0, \Phi_j^N(z) \to e^{-1\lambda\lambda(1-z)}$, a Poisson $(u\lambda t)$ distribution - Thes &-limat: If $r_t \to 0, t \to \infty$ (or $\Lambda_t \to \infty$) while $r_b t \sim r_b \Lambda_t/\lambda = r_1 > 0$. then $\Phi_t^N(z) \to (1 + \kappa(1-z))^{-i}$, a negative binomial distribution again.

4.4.1. The mean of I_t

Letting $I_l = \int_0^t 1\,(N_s = 0)\,ds$ the fraction of the time interval $[0, t]$ free of mutants:

$$
\mathrm{E}\,(I_t) = \int_0^t \Phi_\theta^N(0)ds = \int_0^t (1 + r_b s)^{-\bar{\nu}}\,ds
$$

$$
= \frac{1}{r_b(1-\bar{\nu})}\left((1 + r_b t)^1 - \bar{\nu} - 1\right) \text{ if } \bar{\nu} \neq 1
$$

$$
= \frac{1}{r_b}\log\,(1 + r_b t) \text{ if } \bar{\nu} = 1.
$$

Three enses arises: regime. - if $r_b = \nu\lambda(\nu = 1)$: $\mathrm{E}\,(I_t) = \frac{1}{r_n}\log\,(1 + r_b t) + \frac{1}{r_h}\log t_4$ a logarithmie growth regime: - if $r_b < \nu\lambda(\bar{V} > 1)$: $\mathbf{E}\,(I_l)\underset{t\to\infty}{\sim}\frac{1}{r_n(\bar{\nu}-1)}$: only this constant portion of the positive real line is left uncovered by the mutants. In all cases, $\mathbf{E}\,(I_L)/L, \to_\infty 0$, quite good as well for viral attack protection.

4.4.2. The variance of I_t

Putting $t_2 > t_1$, with

$$
\phi_{t_1,t_2}\,(z_1,, z_2) = \mathrm{E}\left(z_1^{Mt_1} z_2^M\right) = \phi_{t_1}\,(z_1\phi_{t_2-t_1}\,(z_2)).
$$

using (46).

$$
\mathbf{P}\,(N_{t_1} = 0, N_{t_2} = 0) = \Phi_{t_1,t_2}^N(0, 0) =
$$

$$
= \exp -u\lambda\left\{\int_0^{t_1} ds\,(1 - \phi_{t_1\to},(0)) + \int_{t_1}^{t_2} ds\,(1 - \phi_{t_2-s},(0))\right\}
$$

$$
= (1 + r_6 t_1)^{-\bar{\nu}}\,(1 + r_b\,(t_2 - t_1))^{-\bar{\nu}}.
$$

- If $\bar{\nu} \neq 1$, we got

$$\mathrm{E}\left[\left(\int_0^{it} 1\,(N_s = 0)\right)^2\right] = \mathrm{E}\left(/_t^2\right) = \int_0^t \int_{i1}^t \mathrm{P}\,(N_{t_1} = 0, N_{t_{11}} = 0)\, dt_1 dt_2$$

$$= 2\int_0^t dt_1\,(1 + r_b t_1)^{-\bar{D}} \int_{t_1}^t dt_2\,(1 + r_t\,(t_2 - t_1))^{-\bar{t}}$$

$$= \frac{2}{r_0(1 - \bar{\Gamma})}\int_0^t dt_1\,(1 + r_0 t_1) - \bar{p}\left[(1 + r_b\,(t - t_1))^{t - \bar{\nu}} - 1\right]$$

$$= \frac{2}{n_n(1 - \bar{\nu})}\left[h_1 * h_2(t) - \int_0^t w_1 h_1\,(t_1)\right],$$

the first term of which is of convolution type, defining $h_1(t) = (1 + r_b t)^{-\bar{D}}$ and $h_2(t) = (1 + r_k t)^{1 - \bar{\nu}}$. If $\bar{D} \neq 1$, observing and the exponential-integral function $\mathrm{Fi}(x, p) \underset{y \to 0}{\sim} \Gamma(1 - x)p^{x-1}$, we conclude

$$\hat{h}_1(p)\hat{h}_2(p)_{p \to 0}\Gamma\left(1 - ij\Gamma(2 - i)p^{2\pi} - 3\right)$$

$$h_1 * h_2(t) \underset{t \to \infty}{\propto} \frac{\Gamma(1 - \Gamma\Gamma(2 - 4))}{r_b^2} t^{2 - 2\pi}$$

Putting $B := \frac{\Gamma(1 - \bar{\nu})\Gamma(z - \bar{\nu})}{\Gamma(3 - 2\bar{\nu})} = B(1 - \bar{\nu}, 2 - \bar{\nu})$, a beta function, three cases arise: - if $r_b > \nu\lambda(\bar{J} < 1) : \sigma^2\,(I_l) = \mathbf{E}\left(I_t^2\right) - \mathbf{E}\,(I_t)^2 \underset{\sim}{} \frac{1}{\cdot}(2(1 - \bar{\nu})B - 1)t^{2(1 - \bar{\nu})}$.

The standard deviation is of the same order as the mean $\mathrm{E}\,(I_t) \sim \frac{1}{r_0(1 - \bar{m})}t^1$. . - if $T_b < \nu\lambda(\bar{\nu} > 1,$ non-integer $): \sigma^2\,(I_t) \underset{t \to \infty}{\sim} r_t^2(\bar{\nu} - 1)^2$: in this regime, $I_t \underset{t \to \infty}{\{}$ finite non-degenerate rv. - Let us finally consider the case $p = 1$. if $r_b = \nu\lambda(v = 1)$:

$$\mathbf{E}\left(t_i^2\right) = \frac{2}{r_b}\int_0^t dt_1\,(1 + r_b t_1)^{-1}\log\,(1 + r_b\,(t - t_1)) = \frac{t_2}{r_b}h_1 * h_2(t)$$

which is of convolution type, defining $h_1(t) = (1 + r_b t)^{-1}$ and $h_2(t) = \log\,(1 + r_b t)$. We have $\hat{h}_1(p) = \frac{1}{r_h}e^{p/r_0}\,\mathrm{Ei}\,(p/r_h)$ and $\hat{h}_2(p) = \frac{1}{p}e^{p/r_b}\,\mathrm{Ei}\,(p/r_b)$. Using $\mathrm{Fi}(p) \underset{p \to 0+}{\sim} \gamma + \log p$ where γ is the Fuler constant, $\widehat{h}_1(p)\widehat{h}_2(p) \underset{p \to 0+}{\sim} \frac{1}{pm_b}(\log p)^2$, leading, by Laplace inversion, to:

$$\sigma^2\,(I_t) \underset{t \to \infty}{\sim} 2\left(\frac{\log t}{r_b}\right)^2 - \left(\frac{\log t}{r_b}\right)^2 = \left(\frac{\log t}{r_b}\right)^2.$$

The standard deviation is of the same order as the mean $\mathbf{E}\,(I_t) \underset{t \to \infty}{\sim} \frac{1}{r_b}\log t$ as well.

4.4.3. *Covariance of the vacancy process*

Consider the rvs $B_{t_1} := \mathbf{1}\left(N_{t_1} = 0\right)$ and $B_{t_2} := \mathbf{1}\left(N_{t_2} = 0\right), t_2 > t_1 > 0$. With $\tau = t_2 - t_1 > 0$, we have

$$\text{Cov}\left(B_{t_1}, B_{t_1+\tau}\right) = \mathbf{P}\left(N_{t_1} = 0, N_{t_2} = 0\right) - \mathbf{P}\left(N_{t_1} = 0\right)\mathbf{P}\left(N_{t_2} = 0\right)$$
$$= \left(1 + r_b t_1\right)^{-\bar{\nu}}\left[\left(1 + r_b \tau\right)^{-\bar{\nu}} - \left(1 + r_b\left(t_1 + \tau\right)\right)^{-\bar{\nu}}\right] > 0.$$

$\left(B_{t_1}, B_{t_1+\tau}\right)$ are positively correlated and owing to (for each fixed t_1)

$$\text{Cov}\left(B_{t_1}, B_{t_1+\tau}\right) \underset{\tau\,\text{large}}{\sim} \bar{\nu} r_b^{-\bar{\nu}} t_1 \left(1 + r_b t_1\right)^{-\bar{\nu}} \tau^{-(1+\bar{\nu})},$$

there is algebraic power-law decay of the covariances in the shift variable τ, which is integrable near $\tau = \infty$. The covariances are long-ranged.

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